

Question 1. (20 pts.) Suppose we take one observation, X , from the discrete distribution, where

Table 1: Probability mass function of X

x	-2	-1	0	1	2
$P(X = x \theta)$	$(1 - \theta)/4$	$\theta/12$	$1/2$	$(3 - \theta)/12$	$\theta/4$

$$0 < \theta < 1.$$

a) Find a real-valued statistic $T(X)$ with $\mathbb{E}[T(X)] = \theta$ (T is an unbiased estimator of θ , but might take values outside $[0, 1]$).

b) Obtain the maximum likelihood estimator (MLE) $\hat{\theta}(X)$ of θ and show that it is not unique.

c) Is any choice of MLE unbiased?

$$\begin{aligned} \text{a)} \quad T(X) &= 12 \text{ if } X = -1 \\ &= 0 \text{ otherwise} \end{aligned}$$

$$\mathbb{E}_{\theta}(T(X)) = 12 P(X = -1) = 12 \cdot \theta/12 = \theta.$$

$$\text{b)} \quad X = -2, \quad \text{MLE: } \theta = 0$$

$$X = -1, \quad \text{MLE: } \theta = 1$$

$$X = 0, \quad \text{MLE: } \theta \in (0, 1)$$

$$X = 1, \quad \text{MLE: } \theta = 0$$

$$X = 2, \quad \text{MLE: } \theta = 1$$

$$\text{c)} \quad \mathbb{E}(\hat{\theta}(X)) = 1 \times \frac{\theta}{12} + \frac{\hat{\theta}(0)}{2} + 1 \times \frac{\theta}{4} = \theta$$

$$\frac{\hat{\theta}(0)}{2} = \theta - \frac{\theta}{3} = \frac{2\theta}{3}$$

Contradiction

Question 2. (20 pts.) For some real number $\theta \in \Theta = (0, \infty)$, let i.i.d random variables $X_j, j = 1, \dots, n$ all have cumulative distribution function:

$$F_\theta(x) := \mathbb{P}_\theta[X_j \leq x] = \left(\frac{x}{x+1}\right)^\theta, \quad 0 < x < \infty$$

a) Find the maximum likelihood estimator of θ based on observations $\{X_1, \dots, X_n\}$.

b) Find the Fisher information $I_{X_1}(\theta)$ for one observation.

$$a) \quad f_\theta(x) = \theta \left(\frac{x}{x+1}\right)^{\theta-1} \left(\frac{1}{x+1}\right)^2 = \frac{\theta x^{\theta-1}}{(x+1)^{\theta+2}} \quad 0 < x < \infty$$

$$L(\theta | \underline{x}) = \prod_{i=1}^n f_\theta(x_i) = \frac{\theta^n \left(\prod_{i=1}^n x_i\right)^{\theta-1}}{\left[\prod_{i=1}^n (x_i+1)\right]^{\theta+2}}$$

$$\ell(\theta | \underline{x}) = n \log \theta + (\theta-1) \sum_{i=1}^n \log x_i - (\theta+2) \sum_{i=1}^n \log(1+x_i)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^n \log x_i - \sum_{i=1}^n \log(1+x_i) = 0$$

$$\Rightarrow \frac{n}{\theta} = \sum_{i=1}^n \log \left[\frac{1+x_i}{x_i} \right] \Rightarrow \hat{\theta} = \frac{n}{\sum_{i=1}^n \log \left(\frac{1+x_i}{x_i} \right)}$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n}{\theta^2} < 0 \Rightarrow \hat{\theta} \text{ is the MLE.}$$

$$b) \quad I_{X_1}(\theta) = \frac{1}{\theta^2}$$

Question 3. (20 pts.) Let X_1, X_2, \dots, X_n with $n \geq 2$ be i.i.d $\text{Unif}[-\theta, \theta]$ where $\theta > 0$ is an unknown parameter. Find the uniformly minimum variance unbiased estimator (UMVUE) of $\tau(\theta) = \theta + \theta^{-1}$.

$$Y_i = |X_i| \sim U(0, \theta).$$

$$\text{CSS: } Y_{(n)}.$$

$$\begin{aligned} P(Y_{(n)} \leq t) &= \left(\frac{t}{\theta}\right)^n \Rightarrow f_{Y_{(n)}}(t) = n \left(\frac{t}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} \\ &= \frac{n}{\theta^n} t^{n-1} \text{ if } 0 \leq t \leq \theta \\ &= 0, \text{ otherwise} \end{aligned}$$

$$E[Y_{(n)}] = \frac{n}{\theta^n} \int_0^\theta t^n dt = \frac{n}{n+1} \theta$$

$\Rightarrow \frac{n+1}{n} Y_{(n)}$ is an unbiased estimator of θ

$$\begin{aligned} E[Y_{(n)}^{-1}] &= \frac{n}{\theta^n} \int_0^\theta t^{n-2} dt = \frac{n}{\theta^n(n-1)} \theta^{n-1} \\ &= \frac{n}{n-1} \theta^{-1} \end{aligned}$$

$\frac{n-1}{n} Y_{(n)}^{-1}$ is an unbiased estimator of θ^{-1}

$\Rightarrow \frac{n+1}{n} Y_{(n)} + \frac{n-1}{n} Y_{(n)}^{-1}$ is unbiased for $\theta + \theta^{-1}$

Its a function of CSS

$\Rightarrow \frac{1}{n} \left[(n+1) Y_{(n)} + (n-1) Y_{(n)}^{-1} \right]$ is the UMVUE

Question 4. (20 pts.) Let X_1, X_2, \dots, X_n be i.i.d $\text{Pois}(\theta)$ where $\theta > 0$ is an unknown parameter. Find the UMVUE of $\tau(\theta) = (1 + \theta)e^{-\theta}$.

$$P(X=x) = \frac{e^{-\theta} \theta^x}{x!}, \quad x = 0, 1, 2, \dots$$

$$P(X=0) + P(X=1) = e^{-\theta} + \theta e^{-\theta}$$

Unbiased Statistic $S(x) = I(X_1 \leq 1)$.

$$\text{CSS: } T(x) = \sum_{i=1}^n X_i$$

$$\sum_{i=1}^n X_i \sim \text{Pois}(n\theta)$$

$$\text{Form } E[S(x) | T(x)]$$

$$= P(X_1 \leq 1 | \sum_{i=1}^n X_i = t)$$

$$= P(X_1 \leq 1, \sum_{i=1}^n X_i = t)$$

$$\frac{e^{-n\theta} (n\theta)^t}{t!}$$

$$= \frac{P(X_1=0) P(\sum_{i=2}^n X_i = t) + P(X_1=1) P(\sum_{i=2}^n X_i = t-1)}{e^{-n\theta} \frac{(n\theta)^t}{t!}}$$

$$= \frac{e^{-(n-1)\theta} \frac{[(n-1)\theta]^t}{t!} \cdot e^{-\theta}}{e^{-n\theta} \frac{(n\theta)^t}{t!}} +$$

$$\frac{\theta e^{-\theta} \cdot e^{-(n-1)\theta} \frac{[(n-1)\theta]^{t-1}}{(t-1)!}}{e^{-n\theta} \frac{(n\theta)^t}{t!}}$$

$$= \left(\frac{n-1}{n}\right)^t + \frac{t(n-1)^{t-1}}{n^t}$$

$$= \frac{(n-1)^{t-1}}{n^t} \cdot [n + t - 1]$$