Question 1. (20 pts.) Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed as

$$\text{Unif}(\theta, \theta + 1)$$

where $\theta > 0$ is an unknown parameter.

a) (12 points) Is the maximum likelihood estimate (MLE) for θ unique?

b) (8 points) If the answer to part a) is YES, find the MLE. If not, find all possible values of θ for which the likelihood attains its maximum.

a) $\underset{\sim}{X} = (X_1, \ldots, X_n)$, $\underset{\sim}{x} = (x_1, \ldots, x_n)$

$$L(\theta \mid \underset{\sim}{x}) = f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \theta) = \prod_{i=1}^{n} I(\theta \le x_i \le \theta + 1)$$

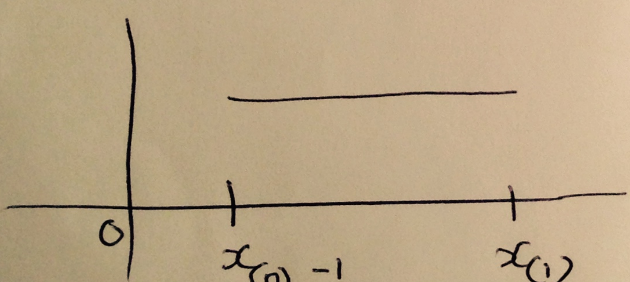
$$= I(x_{(n)} - 1 \le \theta \le x_{(1)})$$

as $x_{(n)} - x_{(1)} \le 1$

Since $P(X_{(n)} - X_{(1)} = 1) = 0$

we have the graph as

either



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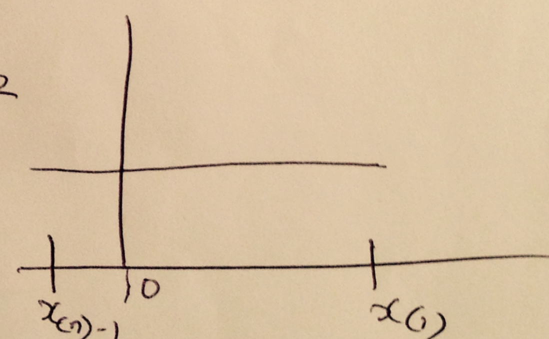


In this case any $\theta \in [X_{(n)} - 1, X_{(1)}]$ is MLE

In this case any $\theta \in [0, X_{(1)}]$ is MLE

MLE NOT UNIQUE

b) For any $\theta \in [\max(X_{(n)} - 1, 0), X_{(1)}]$, the likelihood attains maximum :)

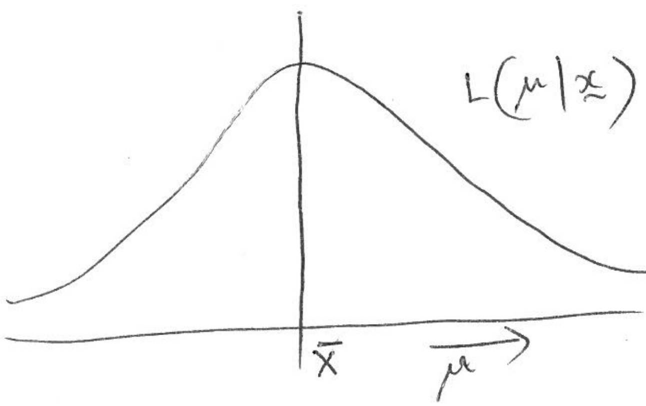
Question 2. (20 pts.) Let $X_1, X_2, \ldots, X_n$ be i.i.d $\mathrm{N}(\mu, 1)$.

a) (10 points) Find the MLE for μ when μ is known to be an integer (belongs to the set $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$).

b) (10 points) Find the MLE for μ when $\mu \in [2, \infty)$.

a) $$L(\mu \mid \underset{\sim}{x}) = f(\underset{\sim}{x} \mid \mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu)^2}$$

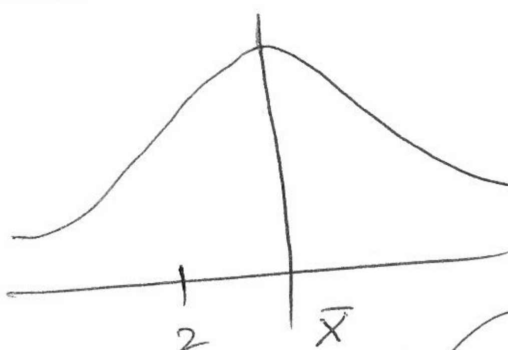
$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2 - \frac{n}{2}(\bar{x}-\mu)^2}$$



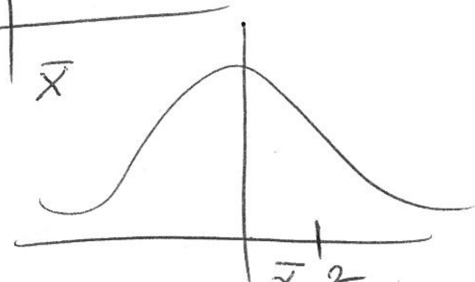
a) If $\mu \in \mathbb{Z}$,

MLE = nearest integer to $\bar{X}$ (can be both Smaller, larger or equal to $\bar{X}$)

b) $\underline{\text{If} \quad \bar{X} > 2}$ MLE = $\bar{X}$



$\underline{\text{If } \bar{X} \leq 2}$ MLE = 2



$\Rightarrow$ MLE = $\max(\bar{x}, 2)$

Question 3. (15 pts.) Let $X_1, \ldots, X_n$ be i.i.d Gamma(α, β) where both α and β are unknown.

a) (10 points) Find the method of moment estimators for α and β.

b) (5 points) Show that the method of moment estimators for α and β are always non-negative.

Let $m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

a) $$EX_1 = \alpha\beta = m_1$$

$$EX_1^2 = \alpha^2\beta^2 + \alpha\beta^2 = \alpha(\alpha+1)\beta^2 = m_2$$

$$\Rightarrow \frac{m_2}{m_1^2} = \frac{\cancel{\alpha}(\alpha+1)\cancel{\beta^2}}{\alpha^{2}\cancel{\beta^2}} = 1 + \frac{1}{\alpha}$$

$$\Rightarrow \frac{1}{\alpha} = \frac{m_2 - m_1^2}{m_1^2} \Rightarrow \hat{\alpha} = \boxed{\frac{m_1^2}{m_2 - m_1^2}}$$

$$\& \quad \hat{\beta} = \frac{m_1}{\hat{\alpha}} = \boxed{\frac{m_2 - m_1^2}{m_1}}$$

b) Sine $$m_2 - m_1^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} X_i - n\bar{X}^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} (X_i - \bar{X})^2\right] \geq 0$$

$\hat{\alpha} \geq 0$

$\hat{\beta} \geq 0$

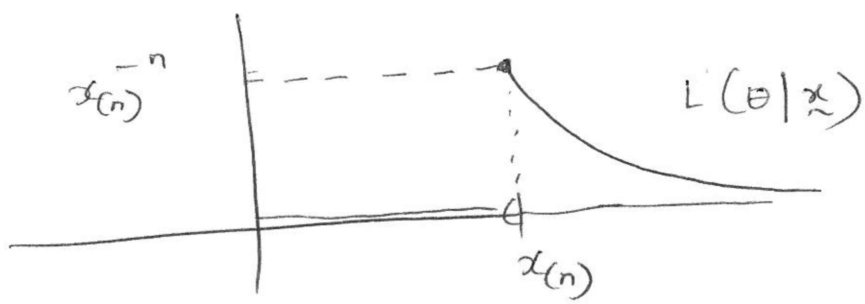
Question 4. (25 pts.) Suppose $X_i, i = 1, \ldots, n$ are i.i.d. samples from Unif$([0, \theta])$, $\theta > 0$.

a) (8 points) Find the MLE for θ^2.

b) (12 points) Show that the MLE obtained in part a) is biased for θ^2.

c) (5 points) Show that for any fixed $\theta > 0$, the *bias* obtained in part b) goes to 0 as $n \to \infty$.

(a) $L(\theta|\underset{\sim}{x}) = f(\underset{\sim}{x}|\theta) = \dfrac{1}{\theta^n} \mathbb{1}[x_{(n)} \le \theta)$



Hence $\hat{\theta}_{MLE} = x_{(n)}$

By invariance principle, $\widehat{\theta^2}_{MLE} = x_{(n)}^2$

(b) $E(X_{(n)}^2) = \displaystyle\int_0^\theta x^2 f_{X_{(n)}}(x)\, dx$

$= \dfrac{n}{\theta^n} \displaystyle\int_0^\theta x^2 \cdot x^{n-1}\, dx$

$= \dfrac{n \, \theta^{n+2}}{(n+2)\, \theta^n} = \dfrac{n\theta^2}{n+2} \neq \theta^2$

$f_{X_{(n)}}(x)$
$= \dfrac{d}{dx} P(X_{(n)} \le x)$
$= \dfrac{d}{dx} \left(\dfrac{x}{\theta}\right)^n$
$= \dfrac{n}{\theta} \left(\dfrac{x}{\theta}\right)^{n-1}$

So $X_{(n)}^2$ is biased for θ^2

(c) Bias: $E(X_{(n)}^2) - \theta^2 = \dfrac{n\theta^2}{n+2} - \theta^2$

$= \dfrac{-2\theta^2}{n+2}$

$\longrightarrow 0$ as $n \to \infty$