STA 5327 Exam 3

April 29, 2016

Name:

FSUID:

Please sign the following pledge and read all instructions carefully before starting the exam.

Pledge: I have neither given nor received any unauthorized aid in completing this exam, and I have conducted myself within the guidelines of the University Honor Code.

Signature: ____

INSTRUCTIONS:

- This is a closed-book, closed-notes exam. However, a formula page is provided at the back.
- Total time is 2 hrs (12:30 P.M to 2:30 P.M.)
- **Show all work**, clearly and in order, if you want to receive full credit. When you use your calculator, explain all relevant mathematics. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- **Answer all the questions in the space provided. You may attach additional sheets if necessary.**
- This test has 4 problems and is worth 80 points. It is your responsibility to make sure that you have all of the problems.
- **Good luck!**

Prob. No.	Max Points	Earned Pts.
1	20	
2	20	
3	20	
4	20	

TOTAL: __________

Question 1. (20 points) Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables with the density function

$$f(x \mid \theta) = \frac{2x}{\theta^2}, \quad 0 < x < \theta.$$

a) (10 points) Find the Crámer-Rao lower bound for any unbiased estimator of $\tau(\theta) = \theta$ using the formula

$$\frac{1}{n\mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(X_1 \mid \theta)\right]^2}.$$

b) (5 points) Find the variance of the unbiased estimator $\frac{3}{2n}\sum_{i=1}^n X_i$.

c) (5 points) Explain the contradictory results in a) and b).

Question 2. (20 points) Let $X_1, \ldots, X_n$ be i.i.d random variables from a regular probability density function $f(x \mid \theta)$ with $\theta = (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p$. Let $I(\theta)$ denote the $p \times p$ information matrix. Show that for any $j \in \{1, \ldots, p\}$, the (j, j)th entry of $I(\theta)^{-1}$ is larger than or equal to the inverse of the (j, j)th entry of $I(\theta)$. Explain the significance of this result in statistical inference of multiple parameters.

Question 3. (20 points) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed from a Beta$(1, \theta)$ distribution for an unknown parameter $\theta > 0$.

a) (5 points) Find the distribution of $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$ using *Central Limit Theorem* given by

$$\sqrt{n}[\bar{X} - \mathbb{E}(X_1)] \xrightarrow{d} \mathrm{N}(0, \mathrm{Var}(X_1)).$$

b) (15 points) A method of moments estimator of θ (you do not need to prove this part) is given by

$$T(X) = \frac{1}{\bar{X}} - 1.$$

Using Delta method, show that $T(X)$ is not an asymptotically efficient estimator.

Question 4. (20 points)

a) (5 points) Consider a test of simple hypotheses

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1$$

based on one observation X from a discrete distribution with probability mass function $f(x \mid \theta)$ for $x = 1, 2, \ldots, 7$. The values of the likelihood function at θ_0 and θ_1 are given in the table below.

x	1	2	3	4	5	6	7
$f(x \mid \theta_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x \mid \theta_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Find the most powerful test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ with level $\alpha = 0.04$. Compute the power for this test.

b) (15 points) Suppose X is a single observation from a population with probability density function given by:

$$f(x \mid \theta) = \theta x^{\theta-1}, \quad 0 < x < 1.$$

where $\theta > 0$ is the parameter of interest. Find the rejection region for the most powerful test of level 0.05, for testing the simple null hypothesis $H_0 : \theta = 3$ against the simple alternative hypothesis $H_1 : \theta = 2$.

Table of Distributions

Distribution	PMF/PDF and Support	Expected Value	Variance	MGF
Bernoulli Bern(p)	$P(X=1)=p$ $P(X=0)=q=1-p$	p	pq	$q+pe^t$
Binomial Bin(n,p)	$P(X=k)=\binom{n}{k}p^kq^{n-k}$ $k\in\{0,1,2,\ldots n\}$	np	npq	$(q+pe^t)^n$
Geometric Geom(p)	$P(X=k)=q^kp$ $k\in\{0,\ 1,\ 2,\ \ldots\}$	q/p	q/p^2	$\frac{p}{1-qe^t},\ qe^t<1$
Negative Binomial NBin(r,p)	$P(X=n)=\binom{r+n-1}{r-1}p^rq^n$ $n\in\{0,\ 1,\ 2,\ \ldots\}$	rq/p	rq/p^2	$(\frac{p}{1-qe^t})^r,\ qe^t<1$
Hypergeometric HGeom(w,b,n)	$P(X=k)=\binom{w}{k}\binom{b}{n-k}/\binom{w+b}{n}$ $k\in\{0,1,2,\ldots,n\}$	$\mu=\frac{nw}{b+w}$	$\left(\frac{w+b-n}{w+b-1}\right)n\frac{\mu}{n}(1-\frac{\mu}{n})$	messy
Poisson Pois(λ)	$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$ $k\in\{0,\ 1,\ 2,\ \ldots\}$	λ	λ	$e^{\lambda(e^t-1)}$
Uniform Unif(a,b)	$f(x)=\frac{1}{b-a}$ $x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Normal $\mathcal{N}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x\,-\,\mu)^2/(2\sigma^2)}$ $x\in(-\infty,\infty)$	μ	σ^2	$e^{t\mu+\frac{\sigma^2t^2}{2}}$
Exponential Expo(λ)	$f(x)=\lambda e^{-\lambda x}$ $x\in(0,\infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gamma Gamma(a,λ)	$f(x)=\frac{1}{\Gamma(a)}(\lambda x)^ae^{-\lambda x}\frac{1}{x}$ $x\in(0,\infty)$	$\frac{a}{\lambda}$	$\frac{a}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^a,\ t<\lambda$
Beta Beta(a,b)	$f(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$ $x\in(0,1)$	$\mu=\frac{a}{a+b}$	$\frac{\mu(1-\mu)}{(a+b+1)}$	messy
Log-Normal $\mathcal{LN}(\mu,\sigma^2)$	$\frac{1}{x\sigma\sqrt{2\pi}}e^{-(\log x-\mu)^2/(2\sigma^2)}$ $x\in(0,\infty)$	$\theta=e^{\mu+\sigma^2/2}$	$\theta^2(e^{\sigma^2}-1)$	doesn't exist
Chi-Square χ^2_n	$\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$ $x\in(0,\infty)$	n	$2n$	$(1-2t)^{-n/2},\ t<1/2$
Student-t t_n	$\frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)}(1+x^2/n)^{-(n+1)/2}$ $x\in(-\infty,\infty)$	0 if $n>1$	$\frac{n}{n-2}$ if $n>2$	doesn't exist