STA 5327 Exam 3

April 28, 2015

Name:

FSUID:

Please sign the following pledge and read all instructions carefully before starting the exam.

Pledge: I have neither given nor received any unauthorized aid in completing this exam, and I have conducted myself within the guidelines of the University Honor Code.

Signature: ____

INSTRUCTIONS:

- This is an closed-book, closed-notes exam. However, 2 formula pages are provided at the back.
- Total time is 2 hrs (10:00 A.M to 12:00 P.M.)
- **Show all work**, clearly and in order, if you want to receive full credit. When you use your calculator, explain all relevant mathematics. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- **Answer all the questions in the space provided. You may attach additional sheets if necessary.**
- This test has 4 problems and is worth 80 points. It is your responsibility to make sure that you have all of the problems.
- **Good luck!**

Prob. No.	Max Points	Earned Pts.
1	20	
2	20	
3	15	
4	25	

TOTAL: __________

Question 1. (20 pts.) Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed as Exponential(θ) where $\theta > 0$ is an unknown parameter. We are interested in estimating θ^2.

a) (12 points) Find the Cramer Rao lower bound for the variance of an unbiased estimator of θ^2.

b) (8 points) Find the Uniformly Minimum Variance Unbiased Estimator (UMVUE) for θ^2. (Hint: Try to find a power of $\bar{X}$ which is an unbiased statistic).

Question 2. (20 pts.) Let $X_1, X_2, \ldots, X_n$ be i.i.d Poisson(λ).

a) (10 points) Find an unbiased estimator for $P(X_1 = 0) = e^{-\lambda}$.

b) (10 points) Find the UMVUE for $e^{-\lambda}$.

Question 3. (15 pts.) Let X_1, X_2 are independently and identically distributed as $f(x \mid \theta)$ where

$$f(x \mid \theta) = \begin{cases} \frac{3x^2}{\theta^3}, & \text{if} \quad 0 < x < \theta \\ 0, & \text{otherwise.} \end{cases}$$

Find the UMVUE for θ. You can assume that $\max(X_1, X_2)$ is a complete sufficient statistic.

Question 4. (25 pts.) Suppose $X_i, i = 1, \ldots, n$ are i.i.d. samples from Unif($[0, \theta]$), $\theta > 0$. Let $T = X_{(n)}$.

a) (8 points) Show that for testing hypothesis $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ for $\theta_1 > \theta_0$, the likelihood Ratio for T is a non-decreasing function.

b) (12 points) Find the most powerful level $\alpha = 0.05$ test for

$$H_0 : \theta \leq 2 \quad \text{versus} \quad H_1 : \theta > 2.$$

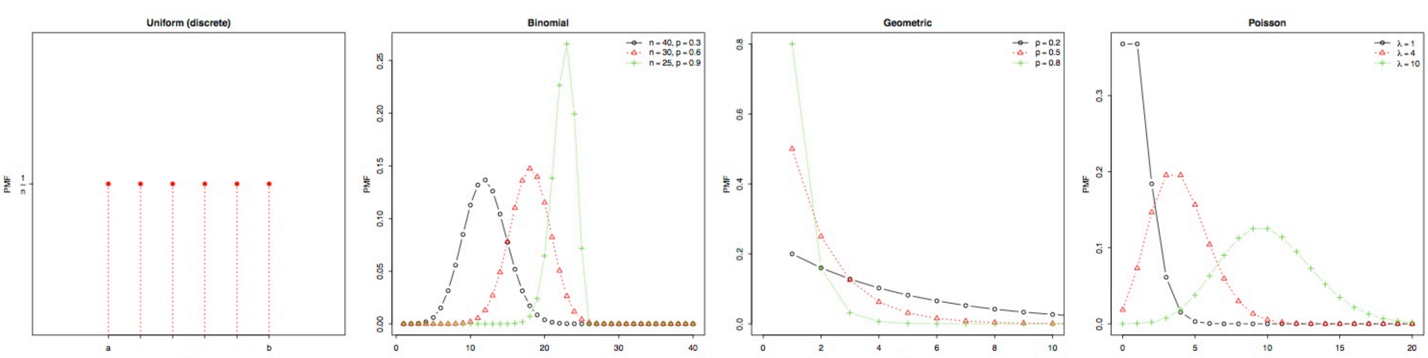
You need to explicitly find the cut-off value for the rejection region.

c) (5 points) Compute and graph the power function for the test in b).

1 Distribution Overview

1.1 Discrete Distributions

	Notation[1]	$F_X(x)$	$f_X(x)$	$\mathbb{E}[X]$	$\mathbb{V}[X]$	$M_X(s)$
Uniform	$\text{Unif}\{a,\ldots,b\}$	$\begin{cases} 0 & x < a \\ \frac{\lfloor x \rfloor - a + 1}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$	$\dfrac{I(a < x < b)}{b-a+1}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a+1)^2-1}{12}$	$\dfrac{e^{as}-e^{-(b+1)s}}{s(b-a)}$
Bernoulli	$\text{Bern}(p)$	$(1-p)^{1-x}$	$p^x(1-p)^{1-x}$	p	$p(1-p)$	$1-p+pe^s$
Binomial	$\text{Bin}(n,p)$	$I_{1-p}(n-x,x+1)$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$(1-p+pe^s)^n$
Multinomial	$\text{Mult}(n,p)$		$\dfrac{n!}{x_1!\ldots x_k!}p_1^{x_1}\cdots p_k^{x_k} \quad \sum_{i=1}^k x_i = n$	np_i	$np_i(1-p_i)$	$\left(\sum_{i=0}^k p_i e^{s_i}\right)^n$
Hypergeometric	$\text{Hyp}(N,m,n)$	$\approx \Phi\left(\dfrac{x-np}{\sqrt{np(1-p)}}\right)$	$\dfrac{\binom{m}{x}\binom{m-x}{n-x}}{\binom{N}{x}}$	$\dfrac{nm}{N}$	$\dfrac{nm(N-n)(N-m)}{N^2(N-1)}$	N/A
Negative Binomial	$\text{NBin}(n,p)$	$I_p(r,x+1)$	$\binom{x+r-1}{r-1}p^r(1-p)^x$	$r\dfrac{1-p}{p}$	$r\dfrac{1-p}{p^2}$	$\left(\dfrac{p}{1-(1-p)e^s}\right)^r$
Geometric	$\text{Geo}(p)$	$1-(1-p)^x \quad x \in \mathbb{N}^+$	$p(1-p)^{x-1} \quad x \in \mathbb{N}^+$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$	$\dfrac{p}{1-(1-p)e^s}$
Poisson	$\text{Po}(\lambda)$	$e^{-\lambda}\sum_{i=0}^{x}\dfrac{\lambda^i}{i!}$	$\dfrac{\lambda^x e^{-\lambda}}{x!}$	λ	λ	$e^{\lambda(e^s-1)}$



[1]We use the notation $\gamma(s,x)$ and $\Gamma(x)$ to refer to the Gamma functions and use $\mathrm{B}(x,y)$ and I_x to refer to the Beta functions

1.2 Continuous Distributions

	Notation	$F_X(x)$	$f_X(x)$	$\mathbb{E}[X]$	$\mathbb{V}[X]$	$M_X(s)$
Uniform	$\text{Unif}(a,b)$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x > b \end{cases}$	$\frac{I(a < x < b)}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{sb}-e^{sa}}{s(b-a)}$
Normal	$\mathcal{N}(\mu,\sigma^2)$	$\Phi(x) = \int_{-\infty}^{x} \phi(t)\,dt$	$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$	μ	σ^2	$\exp\left\{\mu s + \frac{\sigma^2 s^2}{2}\right\}$
Log-Normal	$\ln\mathcal{N}(\mu,\sigma^2)$	$\frac{1}{2} + \frac{1}{2}\,\text{erf}\left[\frac{\ln x - \mu}{\sqrt{2\sigma^2}}\right]$	$\frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}$	$e^{\mu+\sigma^2/2}$	$(e^{\sigma^2}-1)e^{2\mu+\sigma^2}$	
Multivariate Normal	$\text{MVN}(\mu,\Sigma)$		$(2\pi)^{-k/2}\lvert\Sigma\rvert^{-1/2}e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$	μ	Σ	$\exp\left\{\mu^T s + \frac{1}{2}s^T\Sigma s\right\}$
Student's t	$\text{Student}(\nu)$	$I_x\left(\frac{\nu}{2},\frac{\nu}{2}\right)$	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}$	0	0	
Chi-square	χ^2_k	$\frac{1}{\Gamma(k/2)}\gamma\left(\frac{k}{2},\frac{x}{2}\right)$	$\frac{1}{2^{k/2}\Gamma(k/2)}x^{k/2}e^{-x/2}$	k	$2k$	$(1-2s)^{-k/2}\ s < 1/2$
F	$\text{F}(d_1,d_2)$	$I_{\frac{d_1 x}{d_1 x + d_2}}\left(\frac{d_1}{2},\frac{d_1}{2}\right)$	$\frac{\sqrt{\frac{(d_1 x)^{d_1} d_2^{d_2}}{(d_1 x + d_2)^{d_1+d_2}}}}{x\text{B}\left(\frac{d_1}{2},\frac{d_1}{2}\right)}$	$\frac{d_2}{d_2-2}$	$\frac{2d_2^2(d_1+d_2-2)}{d_1(d_2-2)^2(d_2-4)}$	
Exponential	$\text{Exp}(\beta)$	$1-e^{-x/\beta}$	$\frac{1}{\beta}e^{-x/\beta}$	β	β^2	$\frac{1}{1-\beta s}\ (s < 1/\beta)$
Gamma	$\text{Gamma}(\alpha,\beta)$	$\frac{\gamma(\alpha,x/\beta)}{\Gamma(\alpha)}$	$\frac{1}{\Gamma(\alpha)\,\beta^\alpha}x^{\alpha-1}e^{-x/\beta}$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta s}\right)^\alpha\ (s < 1/\beta)$
Inverse Gamma	$\text{InvGamma}(\alpha,\beta)$	$\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{-\alpha-1}e^{-\beta/x}$	$\frac{\beta}{\alpha-1}\ \alpha > 1$	$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)^2}\ \alpha > 2$	$\frac{2(-\beta s)^{\alpha/2}}{\Gamma(\alpha)}K_\alpha\left(\sqrt{-4\beta s}\right)$
Dirichlet	$\text{Dir}(\alpha)$		$\frac{\Gamma\left(\sum_{i=1}^k \alpha_i\right)}{\prod_{i=1}^k \Gamma(\alpha_i)}\prod_{i=1}^k x_i^{\alpha_i-1}$	$\frac{\alpha_i}{\sum_{i=1}^k \alpha_i}$	$\frac{\mathbb{E}[X_i](1-\mathbb{E}[X_i])}{\sum_{i=1}^k \alpha_i + 1}$	
Beta	$\text{Beta}(\alpha,\beta)$	$I_x(\alpha,\beta)$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{s^k}{k!}$
Weibull	$\text{Weibull}(\lambda,k)$	$1-e^{-(x/\lambda)^k}$	$\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k}$	$\lambda\Gamma\left(1+\frac{1}{k}\right)$	$\lambda^2\Gamma\left(1+\frac{2}{k}\right)-\mu^2$	$\sum_{n=0}^{\infty}\frac{s^n\lambda^n}{n!}\Gamma\left(1+\frac{n}{k}\right)$
Pareto	$\text{Pareto}(x_m,\alpha)$	$1-\left(\frac{x_m}{x}\right)^\alpha\ x \geq x_m$	$\alpha\frac{x_m^\alpha}{x^{\alpha+1}}\quad x \geq x_m$	$\frac{\alpha x_m}{\alpha-1}\ \alpha > 1$	$\frac{x_m^\alpha}{(\alpha-1)^2(\alpha-2)}\ \alpha > 2$	$\alpha(-x_m s)^\alpha\Gamma(-\alpha,-x_m s)\ s < 0$