# Solution to Question 4 of Practice problem

a) $X_1, \dots, X_n \sim U(0, \theta)$ $\quad T = X_{(n)}$ : sufficient stat.

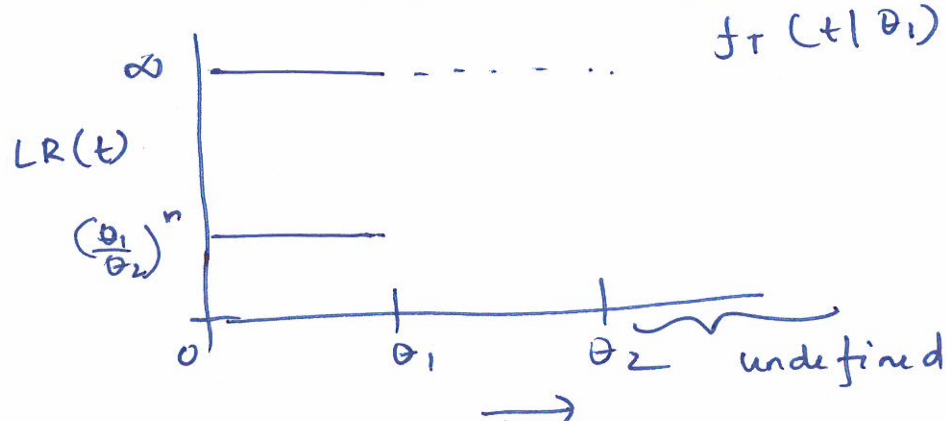
$$P(T \le t) = \left(\frac{t}{\theta}\right)^n \qquad 0 \le t \le \theta$$

$$f_T(t) = \frac{n}{\theta^n} t^{n-1} \qquad 0 \le t \le \theta$$

Fix $\theta_2 > \theta_1 > 0$

$$LR_T(t) = \frac{f_T(t \mid \theta_2)}{f_T(t \mid \theta_1)} = \frac{\frac{n}{\theta_2^n} t^{n-1} \, 1(0 \le t \le \theta_2)}{\frac{n}{\theta_1^n} t^{n-1} \, 1(0 \le t \le \theta_1)}$$

$$= \left(\frac{\theta_1}{\theta_2}\right)^n \frac{1(0 \le t \le \theta_2)}{1(0 \le t \le \theta_1)}$$




$\Rightarrow$ $LR_T(t)$ is a monotone non-decreasing function of $t$.

b) By Karlin-Rubin Theorem, the rejection region of any UMP test will be of the form $T > c$

$$\text{Size} = \sup_{\theta \le 2} P_\theta(T > c) = 0.05$$

$$\Rightarrow \sup_{\theta \le 2} 1 - P_\theta(T \le c) = 0.05$$

$$\Rightarrow \sup_{\theta \le 2} 1 - \left(\frac{c}{\theta}\right)^n = 0.05$$

$$\Rightarrow \quad 1 - \left(\frac{c}{2}\right)^n = 0.05$$

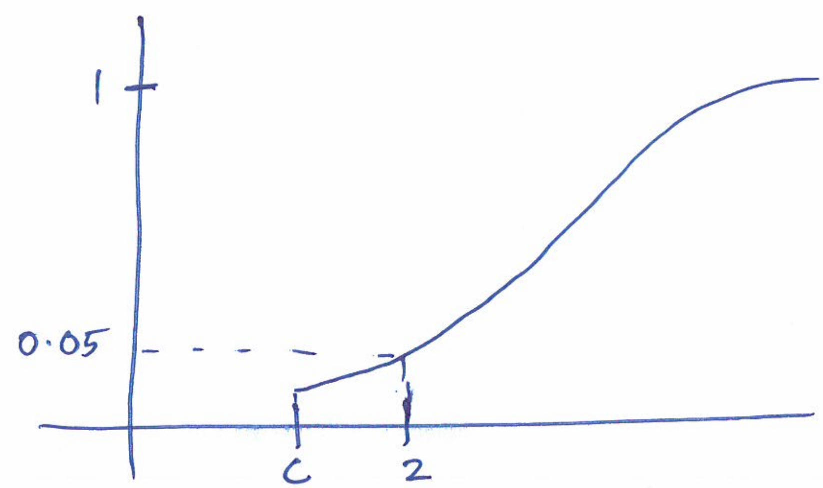
$$\Rightarrow \quad c = 2.(0.95)^{1/n}$$

$\Rightarrow$ Region of Rejection of UMP level 0.05 test is

$$\{ X_{(n)} > 2.(0.95)^{1/n} \}.$$

c) Power function: $\beta(\theta) = P_\theta(T > c)$

$$\Rightarrow \begin{cases} 0 \text{ if } & c > \theta \\ 1 - \left(\frac{c}{\theta}\right)^n & c \le \theta \end{cases}$$




$c = 2.(0.95)^{1/n} < 2$

# Extension to Question 1

Find whether $\cancel{(\bar{X}^2)}$ achieves CRLB $\rightarrow$ $Var\left(\frac{n}{n+1}\bar{X}^2\right)$

$$CRLB = \frac{4\theta^4}{n}$$

$$Var(\bar{X}^2) = Var\left(\frac{1}{n^2}\left(\sum X_i\right)^2\right)$$

$$= \frac{1}{n^4} Var(S^2) \quad \text{where } S = \sum X_i \sim Gamma(n, \theta)$$

$$= \frac{1}{n^4}\left[E(S^4) - E^2(S^2)\right]$$

$$E(S^4) = \frac{1}{\Gamma(n)\theta^n}\int_0^\infty x^4 e^{-x/\theta} x^{n-1}\, dx$$

$$= \frac{1}{\Gamma(n)\theta^n} \boxed{\int_0^\infty x^{n+4-1} e^{-x/\theta}} \rightarrow \text{normalizer of } Gamma(n+4, \theta)$$

$$= \frac{\Gamma(n+4)\,\theta^{n+4}}{\Gamma(n)\,\theta^n} = \frac{(n+4)!\,\theta^{n+4}}{n!\,\theta^n}$$

$$= (n+3)(n+2)(n+1)n\,\theta^4$$

$$E(S^2) = Var(S) + E^2(S)$$

$$= n\theta^2 + n^2\theta^2 = n(n+1)\theta^2$$

$$\Rightarrow Var(\bar{X}^2) = \frac{\theta^4}{n^4}\left[n(n+1)(n+2)(n+3) - n^2(n+1)^2\right]$$

$$= \frac{\theta^4}{n^4} n(n+1)\left[\cancel{n^2} + 5n + 6 - \cancel{n^2} - n\right]$$

$$= \frac{\theta^4}{n^4} n(n+1)(4n+6)$$

$$\mathrm{Var}\left(\frac{n}{n+1}\bar{X}^2\right)$$

$$= \left(\frac{n}{n+1}\right)^2 \frac{\theta^4}{n^4} n(n+1)(4n+6)$$

$$= \frac{\theta^4(4n+6)}{(n+1)n}$$

Clearly $\mathrm{Var}\left(\frac{n}{n+1}\bar{X}^2\right) > \mathrm{CRLB}$

But $\lim_{n\to\infty} \frac{\mathrm{Var}\left(\frac{n}{n+1}\bar{X}^2\right)}{\mathrm{CRLB}}$

$$= \frac{\dfrac{\cancel{\theta^4}(4n+6)}{\cancel{n}(n+1)}}{\dfrac{4\cancel{\theta^4}}{\cancel{n}}}$$

$$= \frac{4n+6}{4(n+1)} \longrightarrow 1$$

as $n \longrightarrow \infty$

So $\left(\frac{n}{n+1}\right)\bar{X}^2$ is asymptotically efficient.