Question 1. (15 points) Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed as Beta$(1,\beta)$ distribution given by

$$f(x \mid \beta) = \beta(1-x)^{\beta-1}, \quad 0 < x < 1$$

where $\beta > 0$ is an unknown parameter. It is easy to show that $\sum_{i=1}^n \log(1-X_i)$ is a complete and sufficient statistic for β and $S = (1-\bar{X})/\bar{X}$ is the method of moments estimator for β, where $\bar{X} = (1/n)\sum_{i=1}^n X_i$. Also, straightforward calculations show that S is not an unbiased estimator. Suppose, for some function g, I found that $S' = g(\bar{X})$ is an unbiased estimator for β. Will S' be the best unbiased estimator of β? Justify your answer.

$\frac{1-\bar{X}}{\bar{X}}$ is unbiased but ~~any~~ the best unbiased estimator should be a function of the complete sufficient statistic $\sum_{i=1}^n \log(1-X_i)$ and not $\bar{X}$. Since $\bar{X}$ is not a function of $\sum_{i=1}^n \log(1-X_i)$, $g(\bar{x})$ is not a function of $\sum_{i=1}^n \log(1-X_i)$

So S' is NOT best unbiased

To prove rigorously that $\bar{X}$ is NOT a function of $\sum_{i=1}^n \log(1-X_i)$

(You'll get full points even if you do not show this)

Consider 2 datasets with $n=2$.

$$X_1 = 1 - e^{-2} \qquad Y_1 = 1 - e^{-1}$$
$$X_2 = 1 - e^{-3} \qquad Y_2 = 1 - e^{-4}$$

$$\log(1-X_1) + \log(1-X_2) = -5 \qquad \log(1-Y_1) + \log(1-Y_2) = -5$$

But $\frac{X_1+X_2}{2} = 1 - \frac{e^{-2}+e^{-3}}{2} \neq \frac{Y_1+Y_2}{2} = 1 - \frac{e^{-1}+e^{-4}}{2}$

Hence $\bar{X}$ is NOT a function of $\sum_{i=1}^n \log(1-X_i)$

Question 2. (20 points) Let $X_1, X_2, \ldots, X_n$ be i.i.d Bernoulli(θ) where $0 < \theta < 1$ is an unknown parameter. Recall that $T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

a) (8 points) Find the best unbiased estimator for $1 - \theta$.

b) (12 points) Find the best unbiased estimator for $(1 - \theta)^2$.

a) $\bar{X}$ is best unbiased for θ. So $1 - \bar{X}$ is best unbiased for $1 - \theta$ (It is a function of the complete sufficient statistic $\bar{X}$)

b) $X_1 + X_2 \sim \text{Binomial}(2, \theta)$

$$P(X_1 + X_2 = 0) = \binom{2}{0} \theta^0 (1-\theta)^{2-0} = (1-\theta)^2$$

$S(X) = I(X_1 + X_2 = 0)$ is an unbiased estimator.

Consider $E\left(S(X) \mid \sum_{i=1}^{n} X_i = t\right)$

$$= \frac{P\left(X_1 + X_2 = 0, \sum_{i=3}^{n} X_i = t\right)}{P\left(\sum_{i=1}^{n} X_i = t\right)}$$

$$= \frac{(1-\theta)^2 \binom{n-2}{t} \theta^t (1-\theta)^{n-2-t}}{\binom{n}{t} \theta^t (1-\theta)^{n-t}}$$

$$= \frac{\binom{n-2}{t}}{\binom{n}{t}} = \frac{(n-2)!}{\cancel{t!}\,(n-2-t)!} \cdot \frac{\cancel{t!}\,(n-t)!}{n!}$$

$$= \frac{(n-t)(n-t-1)}{n(n-1)}$$

Best unbiased estimator for $(1-\theta)^2 = \dfrac{(n - \sum X_i)(n - \sum X_i - 1)}{n(n-1)}$

Question 3. (20 points) Let $X_1, X_2, \ldots, X_n$ be iid $\mathrm{N}(\mu, \sigma^2)$ with unknown mean $\mu \in \mathbb{R}$ and known variance $\sigma^2 > 0$. Recall that $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for μ.

a) (8 points) For fixed and known $t \neq 0$, find the best unbiased estimator of $e^{t\mu}$. (Hint: Try with the moment generating function of $\bar{X}$ (Refer to $M_X(s)$ of $\mathrm{N}(\mu, \sigma^2)$ in the formula page).

b) (12 points) Show that the variance of the best unbiased estimator obtained in a) does not achieve the Crámer-Rao lower bound, but it is asymptotically efficient.

a) $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$E\left(e^{t\bar{X}}\right) = e^{t\mu + \frac{t^2\sigma^2}{2n}}$$

$$\Rightarrow E\left[e^{t\bar{X} - \frac{t^2\sigma^2}{2n}}\right] = e^{t\mu}$$

(*) $\Rightarrow e^{t\bar{X} - \frac{t^2\sigma^2}{2n}}$ is the best unbiased estimator of $e^{t\mu}$

(Since it is a function of CSS $\bar{X}$)

b)

$$f(x_1|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_1-\mu)^2}{2\sigma^2}}$$

$$\frac{\partial}{\partial\mu}\log f = \frac{(x_1-\mu)}{\sigma^2}$$

$$E\left[\left(\frac{\partial}{\partial\mu}\log f\right)^2\right] = \frac{1}{\sigma^2}$$

$$\text{CRLB} = \frac{\left(\frac{\partial}{\partial\mu} e^{t\mu}\right)^2}{n\,I(\theta)} = \frac{t^2 e^{2t\mu}}{\frac{n}{\sigma^2}} = \frac{t^2\sigma^2 e^{2t\mu}}{n}.$$

$$\mathrm{Var}\left(e^{t\bar{X} - \frac{t^2\sigma^2}{2n}}\right)$$

$$= e^{-\frac{t^2\sigma^2}{n}}\left[E\left(e^{2t\bar{X}}\right) - E^2\left(e^{t\bar{X}}\right)\right]$$

$$= e^{-\frac{t^2\sigma^2}{n}}\left[e^{2\mu t + \frac{4t^2\sigma^2}{2n}} - e^{2\mu t + \frac{t^2\sigma^2}{n}}\right]$$

$$= e^{2\mu t}\left[e^{\frac{t^2\sigma^2}{n}} - 1\right]$$

$$\lim_{n\to\infty} \frac{\mathrm{Var}\left(e^{t\bar{X} - \frac{t^2\sigma^2}{2n}}\right)}{\text{CRLB}} = \frac{e^{\frac{t^2\sigma^2}{n}} - 1}{\frac{t^2\sigma^2}{n}} \longrightarrow 1 \qquad \left[\text{Use } \lim_{x\to 0} \frac{e^x - 1}{x} = 1\right]$$

So (*) is asymptotically efficient.

Question 4. (25 points)

a) (10 points) Consider a test of simple hypotheses

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1$$

based on one observation X from a discrete distribution with probability mass function $f(x \mid \theta)$ for $x = 1, 2, \ldots, 7$. The values of the likelihood function at θ_0 and θ_1 are given in the table below.

x	1	2	3	4	5	6	7
$f(x \mid \theta_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x \mid \theta_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Find the most powerful test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ with level $\alpha = 0.04$. Compute the power for this test.

b) (15 points) Suppose X is a single observation from a population with probability density function given by:

$$f(x \mid \theta) = \theta x^{\theta - 1}, \quad 0 < x < 1.$$

where $\theta > 0$ is the parameter of interest. Find the rejection region for the most powerful test of level 0.05, for testing the simple null hypothesis $H_0 : \theta = 3$ against the simple alternative hypothesis $H_1 : \theta = 2$.

a)

x	1	2	3	4	5	6	7
$LR(x)$	6	5	4	3	2	1	$\frac{79}{94}$

$LR(x) = \dfrac{f(x \mid \theta_1)}{f(x \mid \theta_0)}$

Most powerful test with level $\alpha =$ 0.04 has rejection region $\{1, 2, 3, 4\}$.

Power $= 0.06 + 0.05 + 0.04 + 0.03 = 0.18$

b) $LR(x) = \dfrac{f(x \mid 2)}{f(x \mid 3)} = \dfrac{2x^{2-1}}{3x^{3-1}} = \dfrac{2}{3x}$

Rejection region $= \{x : LR(x) > K\} = \left\{x : \dfrac{2}{3x} > K\right\}$

$= \{x : x < K^*\}$

Choose K^* s.t.

$P_{\theta=3}(X < K^*) = 0.05$

$\Rightarrow 3\int_0^{K^*} x^2\,dx = 0.05 \Rightarrow (K^*)^3 = 0.05$

$K^* = (0.05)^{1/3}$

Rejection region for most powerful test is $[X < (0.05)^{1/3}]$

1 Distribution Overview

1.1 Discrete Distributions

	Notation[1]	$F_X(x)$	$f_X(x)$	$\mathbb{E}[X]$	$\mathbb{V}[X]$	$M_X(s)$
Uniform	$\mathrm{Unif}\{a,\ldots,b\}$	$\begin{cases} 0 & x < a \\ \frac{\lfloor x \rfloor - a - 1}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$	$\frac{I(a < x < b)}{b - a + 1}$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2 - 1}{12}$	$\frac{e^{as} - e^{-(b+1)s}}{s(b-a)}$
Bernoulli	$\mathrm{Bern}(p)$	$(1-p)^{1-x}$	$p^x (1-p)^{1-x}$	p	$p(1-p)$	$1 - p + pe^s$
Binomial	$\mathrm{Bin}(n,p)$	$I_{1-p}(n-x, x+1)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$	$(1-p+pe^s)^n$
Multinomial	$\mathrm{Mult}(n,p)$		$\frac{n!}{x_1!\ldots x_k!} p_1^{x_1} \cdots p_k^{x_k} \quad \sum_{i=1}^{k} x_i = n$	np_i	$np_i(1-p_i)$	$\left(\sum_{i=0}^{k} p_i e^{s_i}\right)^n$
Hypergeometric	$\mathrm{Hyp}(N,m,n)$	$\approx \Phi\left(\frac{x - np}{\sqrt{np(1-p)}}\right)$	$\frac{\binom{m}{x}\binom{m-x}{n-x}}{\binom{N}{x}}$	$\frac{nm}{N}$	$\frac{nm(N-n)(N-m)}{N^2(N-1)}$	N/A
Negative Binomial	$\mathrm{NBin}(n,p)$	$I_p(r, x+1)$	$\binom{x+r-1}{r-1} p^r (1-p)^x$	$r\frac{1-p}{p}$	$r\frac{1-p}{p^2}$	$\left(\frac{p}{1-(1-p)e^s}\right)^r$
Geometric	$\mathrm{Geo}(p)$	$1-(1-p)^x \quad x \in \mathbb{N}^+$	$p(1-p)^{x-1} \quad x \in \mathbb{N}^+$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^s}$
Poisson	$\mathrm{Po}(\lambda)$	$e^{-\lambda} \sum_{i=0}^{x} \frac{\lambda^i}{i!}$	$\frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ	$e^{\lambda(e^s - 1)}$

[1] We use the notation $\gamma(s,x)$ and $\Gamma(x)$ to refer to the Gamma functions and use $\mathrm{B}(x,y)$ and I_x to refer to the Beta functions

1.2 Continuous Distributions

	Notation	$F_X(x)$	$f_X(x)$	$\mathbb{E}[X]$	$\mathbb{V}[X]$	$M_X(s)$
Uniform	$\mathrm{Unif}(a,b)$	$\begin{cases} 0 & x<a \\ \frac{x-a}{b-a} & a<x<b \\ 1 & x>b \end{cases}$	$\frac{I(a<x<b)}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{sb}-e^{sa}}{s(b-a)}$
Normal	$\mathcal{N}(\mu,\sigma^2)$	$\Phi(x)=\int_{-\infty}^{x}\phi(t)\,dt$	$\phi(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$	μ	σ^2	$\exp\left\{\mu s+\frac{\sigma^2 s^2}{2}\right\}$
Log-Normal	$\ln\mathcal{N}(\mu,\sigma^2)$	$\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left[\frac{\ln x-\mu}{\sqrt{2\sigma^2}}\right]$	$\frac{1}{x\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}$	$e^{\mu+\sigma^2/2}$	$(e^{\sigma^2}-1)e^{2\mu+\sigma^2}$	
Multivariate Normal	$\mathrm{MVN}(\mu,\Sigma)$		$(2\pi)^{-k/2}\lvert\Sigma\rvert^{-1/2}e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$	μ	Σ	$\exp\left\{\mu^T s+\frac{1}{2}s^T\Sigma s\right\}$
Student's t	$\mathrm{Student}(\nu)$	$I_x\left(\frac{\nu}{2},\frac{\nu}{2}\right)$	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}$	0	0	
Chi-square	χ^2_k	$\frac{1}{\Gamma(k/2)}\gamma\left(\frac{k}{2},\frac{x}{2}\right)$	$\frac{1}{2^{k/2}\Gamma(k/2)}x^{k/2}e^{-x/2}$	k	$2k$	$(1-2s)^{-k/2}\ s<1/2$
F	$\mathrm{F}(d_1,d_2)$	$I_{\frac{d_1 x}{d_1 x+d_2}}\left(\frac{d_1}{2},\frac{d_1}{2}\right)$	$\frac{\sqrt{\frac{(d_1x)^{d_1}d_2^{d_2}}{(d_1x+d_2)^{d_1+d_2}}}}{x\mathrm{B}\left(\frac{d_1}{2},\frac{d_1}{2}\right)}$	$\frac{d_2}{d_2-2}$	$\frac{2d_2^2(d_1+d_2-2)}{d_1(d_2-2)^2(d_2-4)}$	
Exponential	$\mathrm{Exp}(\beta)$	$1-e^{-x/\beta}$	$\frac{1}{\beta}e^{-x/\beta}$	β	β^2	$\frac{1}{1-\beta s}\ (s<1/\beta)$
Gamma	$\mathrm{Gamma}(\alpha,\beta)$	$\frac{\gamma(\alpha,x/\beta)}{\Gamma(\alpha)}$	$\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta s}\right)^\alpha\ (s<1/\beta)$
Inverse Gamma	$\mathrm{InvGamma}(\alpha,\beta)$	$\frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{-\alpha-1}e^{-\beta/x}$	$\frac{\beta}{\alpha-1}\ \alpha>1$	$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)^2}\ \alpha>2$	$\frac{2(-\beta s)^{\alpha/2}}{\Gamma(\alpha)}K_\alpha\left(\sqrt{-4\beta s}\right)$
Dirichlet	$\mathrm{Dir}(\alpha)$		$\frac{\Gamma\left(\sum_{i=1}^k\alpha_i\right)}{\prod_{i=1}^k\Gamma(\alpha_i)}\prod_{i=1}^k x_i^{\alpha_i-1}$	$\frac{\alpha_i}{\sum_{i=1}^k\alpha_i}$	$\frac{\mathbb{E}[X_i](1-\mathbb{E}[X_i])}{\sum_{i=1}^k\alpha_i+1}$	
Beta	$\mathrm{Beta}(\alpha,\beta)$	$I_x(\alpha,\beta)$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{s^k}{k!}$
Weibull	$\mathrm{Weibull}(\lambda,k)$	$1-e^{-(x/\lambda)^k}$	$\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k}$	$\lambda\Gamma\left(1+\frac{1}{k}\right)$	$\lambda^2\Gamma\left(1+\frac{2}{k}\right)-\mu^2$	$\sum_{n=0}^{\infty}\frac{s^n\lambda^n}{n!}\Gamma\left(1+\frac{n}{k}\right)$
Pareto	$\mathrm{Pareto}(x_m,\alpha)$	$1-\left(\frac{x_m}{x}\right)^\alpha\ x\geq x_m$	$\alpha\frac{x_m^\alpha}{x^{\alpha+1}}\ x\geq x_m$	$\frac{\alpha x_m}{\alpha-1}\ \alpha>1$	$\frac{x_m^\alpha}{(\alpha-1)^2(\alpha-2)}\ \alpha>2$	$\alpha(-x_m s)^\alpha\Gamma(-\alpha,-x_m s)\ s<0$