Question 1. (10 pts.) An IRS agent receives a batch of 15 tax returns that were flagged by computer for possible tax evasions. Suppose, unknown to the agent, 3 of these returns have illegal deductions and the other 12 are in good standing. If the agent randomly selects 4 of these returns for audit, what is the probability that:

a) All three of the returns that have illegal deductions are selected?

b) At least 2 have illegal deductions?

a) P(all three returns with illegal deductions are selected) $= \dfrac{\binom{3}{3}\binom{12}{1}}{\binom{15}{4}}$

b) P(at least 2 have illegal deductions)

= P(exactly 2 have illegal deduction) + P(exactly 3 have illegal deductions

$$= \frac{\binom{3}{2}\binom{12}{2}}{\binom{15}{4}} + \frac{\binom{3}{3}\binom{12}{1}}{\binom{15}{4}}$$

Question 2. (10 pts.) On a Monday afternoon, 167 customers will be observed during check-out and the number paying by card will be recorded. Records from the store suggest that 58% of customers pay by card. Approximate the following probabilities using the normal approximation to the binomial. Remember to use continuity correction.

a) Fewer than 101 will pay by card.

b) Between 88 and 100 (inclusive) will pay by card.

X denotes the r.v which is the number of persons paying by card

$$X \sim \text{bin}(167, 0.58)$$

a) $$P(X < 101) = P(X \leq 100)$$

$$\approx P\left(\frac{X - 0.58 \times 167}{\sqrt{0.58 \times 0.42 \times 167}} \leq \frac{100.5 - 0.58 \times 167}{\sqrt{0.58 \times 0.42 \times 167}}\right)$$

$$= \Phi(0.5707) = 0.7159$$

b) $$P(88 \leq X \leq 100)$$

$$\approx P\left(\frac{88 - 0.5 - 0.58 \times 167}{\sqrt{0.58 \times 0.42 \times 167}} \leq \frac{X - 0.58 \times 167}{\sqrt{0.58 \times 0.42 \times 167}} \leq \frac{100.5 - 0.58 \times 167}{\sqrt{0.58 \times 0.42 \times 167}}\right)$$

$$= \Phi(0.5707) - \Phi(-1.4675)$$

$$= 0.6448$$

Question 3. (10 pts.) Four equally qualified undergraduate students apply for jobs at the university bookstore.

Student	Sex	Year
1	M	Junior
2	M	Junior
3	F	Sophomore
4	M	Sophomore

Two students will be chosen at random to receive jobs. Let

A = [Selected students are of the same sex]
B = [Selected students are of the same year]

a) Find the probability of $A \cup B$. (*Hint: List all possible outcomes. Then draw a Venn diagram.*)

b) Are A and B independent? Explain why or why not.

c) Find the probability of $\bar{A} \cap \bar{B}$. ($\bar{A}$ is the complement of A)

a) All possible outcomes $\{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

b) $P(A \cap B) = \frac{1}{6}$, $P(A) \cdot P(B) = \frac{3 \cdot 2}{6 \cdot 6} = \frac{1}{6}$

A and B are independent.

c) $P(A^c \cap B^c) = 1 - P(A \cup B)$
$= P((A \cup B)^c) = 1 - \frac{2}{3} = 1/3$

Question 4. (20 pts) A fair die is tossed and its outcome is denoted by X. After that X independent fair coins are tossed and the number of heads obtained is denoted by Y. Compute

a) $P(Y = 4)$

b) $P(X = 5 \mid Y = 4)$

c) $E(Y)$

d) $E(XY)$

a) For $k = 1, 2, \ldots, 6$, conditionally on $X = k$, Y has the binomial distribution with parameters k and $1/2$. So

$$P(Y=i \mid X=k) = \begin{cases} \binom{k}{i} 2^{-k}, & 0 \le i \le k \\ 0, & i > k \end{cases}$$

So by law of total probability

$$P(Y=4) = \sum_{k=1}^{6} P[Y=4 \mid X=k]\, P(X=k)$$

$$= \frac{1}{6}\left(2^{-4} + \binom{5}{4} 2^{-5} + \binom{6}{4} 2^{-6}\right) = \frac{29}{384}$$

b) By Bayes' formula,

$$P(X=5 \mid Y=4) = \frac{P(Y=4 \mid X=5) \cdot P(X=5)}{P(Y=4)}$$

$$= \frac{10}{29}$$

c) Since $E(Y \mid X=k) = \frac{k}{2}$ (expectation of binomial with $n=k$ and $p = 1/2$)

$$E(Y) = E(E(Y \mid X))$$

$$= \sum_{k=1}^{6} E(Y \mid X=k)\, P(X=k) = \frac{1}{6} \sum_{k=1}^{6} \frac{k}{2} = \frac{7}{4}$$

d) $$E(XY) = E(E(XY \mid X)) = \sum_{k=1}^{6} E(kY \mid X=k)\, P(X=k)$$

$$= \frac{1}{6} \sum_{k=1}^{6} k \cdot E(Y \mid X=k) = \frac{1}{6} \sum_{k=1}^{6} k \cdot \frac{k}{2} = \frac{91}{12}$$

Question 5. (10 pts) There are 10 balls in an urn numbered 1 through 10. You randomly select 3 of those balls. Let the random variable Y denote the maximum of the three numbers on the extracted balls. Find the probability mass function of Y . You should simplify your answer to a fraction that does not involve binomial coefficients. Then calculate $P[Y \geq 7]$.

The random variable Y can take values in the set $\{3, 4, \cdots, 10\}$. for any i, the triplet resulting in Y attaining the value i must consist of the ball numbered i and a pair of balls with lower numbers.

$$\text{So } P(Y=i) = \frac{\binom{i-1}{2}}{\binom{10}{3}} = \frac{\frac{(i-1)(i-2)}{2}}{\frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}} = \boxed{\frac{(i-1)(i-2)}{240} \quad i \geq 3,\ i \leq 10}$$

$$\text{So } P(Y \geq 7) = \frac{6\cdot 5}{240} + \frac{7\cdot 6}{240} + \frac{8\cdot 7}{240} + \frac{9\cdot 8}{240}$$

$$= \frac{1}{240}\left[30 + 42 + 56 + 72\right)$$

$$= \frac{200}{240} = \boxed{\frac{5}{6}}$$

Question 6. (20 pts) The bid that a competitor makes on a real estate property is estimated to be somewhere between 0 and 3 million dollars. Specifically, the bid X is viewed to be a continuous random variable with density function.

$$f(x) = \begin{cases} c(9 - x^2), 0 < x < 3 \\ 0, \textit{ otherwise} \end{cases}$$

You make a bid (without knowing the competitors bid). The higher of the two bids win.

a) Find the value of c that makes $f(x)$ a legitimate density function?

b) Find the cumulative distribution function, $F(x)$. Use the cumulative distribution to determine the probability that you lose the bid if you make a bid of 2 million? 1 million?

c) Find the expected value and standard deviation for the competitors bid. What is the probability that the competitors bid is within one standard deviation of the mean?

d) How much should you bid so that you have a 90% chance of winning?

a) $c \int_0^3 (9-x^2)\,dx = 1 \Rightarrow c\left[9x - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow c\,[27-9] = 1$

$\Rightarrow c = \frac{1}{18}$

b) $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^x \frac{9-t^2}{18}\,dt = \frac{x}{2} - \frac{x^3}{54}$ for $0 < x < 3$

$F(x) = 0$ for $x < 0$ and 1 for $x > 3$

Hence $P(X>2) = 1-F(2) = 1 - \left(\frac{2}{2} - \frac{8}{54}\right) = \frac{8}{54}$

$P(X>1) = 1-F(1) = 1 - \left(\frac{1}{2} - \frac{1}{54}\right) = \frac{28}{54}$

c) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^3 x.\frac{(9-x^2)}{18}\,dx = \left[\frac{x^2}{4} - \frac{x^4}{72}\right]_0^3$

$= 9/8 = 1.125$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^3 x^2 \frac{(9-x^2)}{18}\,dx = 1.8$

$V(X) = 1.8 - \left(\frac{9}{8}\right)^2 = 0.534$. Stand. dev. $= \sqrt{0.534}$

$= 0.731$

Finally 1 standard dev. of mean is $1.125 - 0.731$ to $1.125 + 0.731 = [0.394, 1.856]$

$F(1.856) - F(0.394) = 0.8096 - 0.1959 = 0.6137$

d) We need to find x s.t $F(x) = .9$ (just say this ; no need to sho

Question 7. (20 pts) Which of the following statements are always true? Give brief reasons.

a) $E(XY) = E(X)E(Y)$ $\longrightarrow$ NOT ALWAYS TRUE (See example)

b) $E(X+Y) = E(X) + E(Y)$ $\longrightarrow$ ALWAYS TRUE

c) $E(\frac{X}{Y}) = \frac{E(X)}{E(Y)}$ $\longrightarrow$ NOT ALWAYS TRUE (see example)

a) $X \sim \text{bern}\left(\frac{1}{2}\right), \quad Y = X+1$

$$E(XY) = E(X^2) + E(X)$$

$$= V(X) + E^2(X) + E(X)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$$

$$E(X) = \frac{1}{2}$$

$$E(X+1) = \frac{3}{2} = 3/4$$

c) $Y = \begin{cases} 1 & \text{w.p } 1/2 \\ 2 & \text{w.p } 1/2 \end{cases}$

$$X = 1$$

$$E\left(\frac{X}{Y}\right) = E\left(\frac{1}{Y}\right) = 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$$

$$\frac{E(X)}{E(Y)} = \frac{1}{1 \cdot 1/2 + 2 \cdot 1/2} = 2/3 \quad = 3/4$$