Homework 0 (due Sep 20, 2016)

Problem 1: [Outlier problem] Generate 100 observations from the mixture $0.95N(0,1) + 0.025N(3,1) + 0.025N(-3,1)$. This is to mimic a setting where a small fraction of the data are outliers: 95% of the data have a $N(0,1)$ distribution while the remaining 5% are outliers, which we assume are from a $N(\pm 3, 1)$ distribution. Assume a normal $N(\mu, \tau^{-1})$ model for the data, with priors $\mu \mid \tau \sim N(0, \tau_0^{-1}\tau^{-1})$ with $\tau_0 = 0.1$ and $\tau \sim \text{Gamma}(0.5, 0.5)$. Report a 95% credible interval for μ. Write down the predictive distribution for a future observation $x \mid x_1, \ldots, x_{100}$.

Problem 2: [Polling example] 1447 adults were interviewed prior to the presidential election in 1988, when 727 favored George Bush, 583 favored Michael Dukakis and 137 had no opinion or favored other candidates. Let θ_1 and θ_2 denote the population proportions in favor of Bush and Dukakis respectively. With a uniform prior on the simplex for (θ_1, θ_2), find the posterior probability $P(\theta_1 > \theta_2 \mid \text{data})$, clearly indicating how you calculate the quantity.

Problem 3: [Psychokinesis example:] Recall $x \sim \text{Binomial}(n, \theta)$ with $n = 104,490,000$ and $x = 52,263,470$. We want to test the hypothesis $H_0 : \theta = 1/2$ versus $H_1 : \theta \neq 1/2$. Assume equal prior probabilities on the two hypothesis and consider $\text{Beta}(\alpha, \alpha)$ prior on θ under H_1. Draw a graph plotting the posterior probability $P(\theta = 0.5 \mid x)$ versus α. [It should be clear from the graph what values of α lead to $P(\theta = 0.5 \mid x) = 0.8, 0.7, 0.6, 0.5$ etc] Briefly explain your findings.

Problem 4: Suppose $x \mid \theta \sim \text{Binomial}(n, \theta)$ and we want to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Consider the following class of priors on $[0, 1]$:

$$\mathcal{G} = \left\{ g : \int_0^{\theta_0} g(\theta) d\theta = 1/2 \right\}.$$

Clearly, $\mathcal{G}$ consists of all densities g on $[0, 1]$ with median θ_0. Show that

$$\sup_{g \in \mathcal{G}} \int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x} g(\theta) d\theta \leq \frac{1}{2} \left[\binom{n}{x} \theta_0^x (1-\theta_0)^{n-x} + \binom{n}{x} \left(\frac{x}{n}\right)^x \left(1 - \frac{x}{n}\right)^{n-x} \right].$$

Use this to bound the worst case (over $\mathcal{G}$) posterior probability of H_0 in Problem 3.

Problem 5: [Laplace as variance mixture of normal:] Let $\text{DE}(\tau)$ denote the double exponential (or Laplace) distribution with density $f(x) = (2\tau)^{-1} e^{-|x|/\tau}, x \in \mathbb{R}$. Let $\text{Exp}(\lambda)$ denote an exponential distribution with density $\lambda e^{-\lambda x}$ for $x \in (0, \infty)$. Show that if $\theta \mid \psi \sim N(0, \psi\sigma^2)$ and $\psi \sim \text{Exp}(1/2)$, then $\theta \sim \text{DE}(\sigma)$. [**Hint**: you may find the following fact useful. For any $a, b, p > 0$,

$$f(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{(p-1)} e^{-(ax + b/x)/2}, \quad x \in (0, \infty)$$

is a probability density, where K_p is the modified Bessel function of the second kind. Analytic expressions for K_p are available when $p = m/2$ for an integer m. In particular, $K_{1/2}(z) = \sqrt{\pi/(2z)} e^{-z}$.]

Problem 6: Suppose $\bar{x} \sim N(\theta, \sigma^2/n)$ and we wish to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Set $\pi(\theta) = 0.5\delta_{\theta_0}(\theta) + 0.5\mathrm{N}(\theta_0, \sigma^2)$ where $\delta_{\theta_0}(\theta)$ denotes the Dirac delta measure at θ_0. Show that the Bayes factor in favor of the alternative, $B_{10} = (1+n)^{-1/2} \exp\left[\frac{t^2}{2}\frac{n}{n+1}\right]$ where $t = \sqrt{n}|\bar{x} - \theta_0|/\sigma$. Suppose the true $\theta = \theta^*$, with $\theta^* \neq \theta_0$. Show that $B_{10} \xrightarrow{P} \infty$, i.e., for any $M > 0$, $P_{\theta^*}(B_{10} > M) \to 1$ as $n \to \infty$.

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