Hw0 for 6448 (due Jan 21) (Debdeep Pati)

1. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. Further assume that X and Y are L^2-integrable random variables on $(\Omega, \mathcal{F}, P)$ and that Y is $\mathcal{G}$-measurable.
   (a) Prove $\mathrm{Var}(X \mid \mathcal{G}) \leq \mathrm{E}((X-Y)^2 \mid \mathcal{G})$ almost surely P, and use this fact to verify $\mathrm{E}(\mathrm{Var}(X \mid \mathcal{G})) \leq \mathrm{Var}(X)$.
   (b) Assume f is a contraction on $\mathbb{R}$ meaning that $f : \mathbb{R} \to \mathbb{R}$ and
   $$|f(x) - f(y) \leq |x - y|, \quad \text{for all} \quad x, y \in \mathbb{R}$$
   with strict inequality for some pair $x, y \in \mathbb{R}$. Prove that $f(X)$ is L^2-integrable and $\mathrm{Var}(f(X) \mid \mathcal{G}) \leq \mathrm{Var}(X \mid \mathcal{G})$ almost surely P.
   (c) Now assume that $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}$, where $\mathcal{G} = \sigma(\cup_n \mathcal{G}_n)$. Suppose $Y = \mathrm{E}(X \mid \mathcal{G})$ and let $Y_n = \mathrm{E}(X \mid \mathcal{G}_n)$. Assume $Y_n \to Y$ almost surely, as $n \to \infty$, and show that $\mathrm{E}((Y - Y_n)^2) \to 0$.

2. A mean zero real random variable X is called sub-Gaussian if there exists a constant $C > 0$ such that
   $$\mathrm{E}(e^{tX}) \leq e^{Ct^2} \quad \text{for all} \quad t \in \mathbb{R}. \tag{1}$$
   (a) Determine whether the following random variables are sub-Gaussian:
      i. $X \sim \mathrm{N}(0, 1)$.
      ii. $\Theta \sim \mathrm{Uniform}(-1, 1)$.
      iii. $W = X_1 X_2$, where X_1 and X_2 are independent $\mathrm{N}(0, 1)$ random variables.
   (b) For any real random variable Z, prove
   $$\mathrm{P}(Z \geq t) \leq \inf_{\lambda > 0} e^{-\lambda t} \mathrm{E}(e^{\lambda Z}), \quad \text{for all} \quad t \geq 0.$$
   Use this inequality (or other) to prove, for a sub-Gaussian random variable X satisfying (1), that
   $$\mathrm{P}(X \geq t) \leq e^{-Kt^2}, \quad \text{for all} \quad t \geq 0.$$
   where K is a constant depending only on C.
   (c) Let $X_1, \ldots, X_n$ be independent sub-Gaussian random variables satisfying (1) with the same constant C. Let $\bar{X} = n^{-1} \sum_{i=1}^n X_i$. Prove that
   $$\mathrm{P}(\bar{X} \geq t) \leq e^{-Knt^2}, \quad \text{for all} \quad t \geq 0, n \geq 1.$$
   for a constant K depending on C but not on n and t.