Hw0 Solutions

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1(a)

$$
\begin{aligned}
E((X-Y)^2|\mathcal{G}) &= E((X-E(X|\mathcal{G})+E(X|\mathcal{G})-Y)^2|\mathcal{G}) && (1)\\
&= Var(X|\mathcal{G}) + E((E(X|\mathcal{G})-Y)^2|\mathcal{G}) - 2E((X-E(X|\mathcal{G}))(E(X|\mathcal{G})-Y)|\mathcal{G}) && (2)\\
&= Var(X|\mathcal{G}) + (E(X|\mathcal{G})-Y)^2 && (3)\\
&\geq Var(X|\mathcal{G}) && (4)
\end{aligned}
$$

Justifications:

- All expansions are OK because X and Y are both L^2-integrable.
- (1) to (2): Expand the square and use the linearity of conditional expectation.
- (2) to (3): $E((E(X|\mathcal{G})-Y)^2|\mathcal{G}) = (E(X|\mathcal{G})-Y)^2$ because both $E(X|\mathcal{G})$ and Y are G-easurable. For the same reason, we have

$$
\begin{aligned}
&E((X-E(X|\mathcal{G}))(E(X|\mathcal{G})-Y)|\mathcal{G})\\
&= (E(X|\mathcal{G})-Y)E(X-E(X|\mathcal{G})|\mathcal{G})\\
&= (E(X|\mathcal{G})-Y)(E(X|\mathcal{G})-E(X|\mathcal{G})) = 0
\end{aligned}
$$

- (3) to (4): $(E(X|\mathcal{G})-Y)^2$ is nonnegative almost surely P. Finally, apply the inequality from (a) with $E(X)$ in place of Y.

(b) Define $Z = E(X|\mathcal{G})$(almost surely P). By holding y constant in the definition of the contraction, we see that $f(X)$ is L^2-integrable (and analogously for $f(Z)$).Then,

$$\begin{aligned} Var(f(X)|\mathcal{G}) &\leq E((f(X)-f(Z))^2|\mathcal{G}) && (5)\\ &\leq E((X-Z)^2|\mathcal{G}) && (6)\\ &= E((X-E(X|\mathcal{G}))^2|\mathcal{G}) = Var(X|\mathcal{G}) && (7)\end{aligned}$$

Justifications:

- (5):From (a) with $f(X)$in place of $X$and $f(Z)$in place of Y.
- (5) to (6): Because f is a contraction.
- (6) to (7): Definitions.

(c) First,

$$0 \leq E((Y-Y_n)^2) = E(Y^2) - E(Y_n^2), \tag{8}$$

since

$$E(YY_n) = E(E(YY_n)|\mathcal{G}_n) = E(Y_n E(E(X|\mathcal{G})|\mathcal{G}_n)) = E(Y_n E(X|\mathcal{G}_n)) = E(Y_n^2). \tag{9}$$

Second, by Fatou's lemma,

$$E(Y^2) \geq \lim_{n\to\infty} E(Y_n^2) \geq E(\lim_{n\to\infty} Y_n^2) = E(Y^2) \tag{10}$$

4(a) i. Yes, since $E(e^{tX}) = e^{t^2/2}$ for all t.
ii. Yes, since

$$E(e^{t\Theta}) = \frac{e^t - e^{-t}}{2t} = 1 + \frac{t^2}{3!} + \frac{t^4}{5!} + \cdots \leq 1 + t^2 + \frac{t^4}{2!} + \cdots = e^{t^2}$$

iii. No. For example with $t = 1$, we find,

$$E(e^{X_1X_2}) = E(Ee^{X_1X_2}|X_1)) = E(e^{X_1^2}/2) = \infty.$$

(b) Fix $\lambda \geq 0$. For any $t \geq 0$, we have

$$P(Z \geq t) \leq P(e^{\lambda Z} \geq e^{\lambda t}) \leq e^{-\lambda t} E(e^{\lambda Z}),$$

where the second inequality follows from Markov's inequality. Since the left hand side does not depend on λ, we get

$$P(Z \geq t) \leq \inf_{\lambda \geq 0} (e^{-\lambda t} E(e^{\lambda Z})).$$

if X is sub-Gaussian, we then have

$$P(X \geq t) \leq \inf_{\lambda \geq 0} (e^{-\lambda t} e^{C\lambda^2} \geq e^{-t^2/(4C)}),$$

since the infimum is attained at $\lambda = t/(2C)$.

(c) Since $X_1, ..., X_n$ are independent,

$$E(e^{t \sum_{i=1}^{n} X_i/n}) = \prod_{i=1}^{n} E(e^{(t/n)X_i}) \leq e^{Ct^2/n}.$$

Applying part (b), but replacing C with C/n, we therefore have,

$$P(\bar{X} \geq t) \leq e^{-nt^2/(4C)}.$$