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Warm-up problems from STA 5326 Name: ....

STA 5327 (Instructor: Debdeep Pati) **FSUID:** ....

Work your problems in the space provided. Show all work clearly. Justify your answers. Draw a box around your final answer.

1. Suppose X and Y are independent exponentially distributed random variables with parameter $\beta > 0$ (and pdf $f(x) = \beta^{-1}\exp(-x/\beta)$ for $x > 0$). Consider transformations $V = X+Y$ and $W = X/(X+Y)$.

(a) Find the joint distribution of V and W.

Set $V = X + Y$, $W = X/(X+Y)$ $\Rightarrow$ $X = VW$, $Y = V(1-W)$ $\Rightarrow$ $J(v,w) = \begin{pmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{pmatrix} = \begin{pmatrix} w & v \\ 1-w & -v \end{pmatrix}$

$|J(v,w)| = |-wv - v + wv| = v$

$f_{X,Y}(x,y) = \beta^{-2} e^{-\frac{(x+y)}{\beta}}, \quad x, y > 0$

$$\boxed{f_{V,W}(v,w) = v\beta^{-2} e^{-v/\beta}, \quad v \in (0,\infty), \quad w \in (0,1)}$$

(b) Specify the marginal distributions of V and W. Are V and W independent?

$$f_{V,W}(v,w) = \underbrace{\beta^{-2} v e^{-v/\beta} 1_{(0,\infty)}(v)}_{f_V(v)} \cdot \underbrace{1_{(0,1)}(w)}_{f_W(w)}$$

Since joint density splits as product of marginals, V, W are independent &

$$\boxed{\begin{array}{l} V \sim \text{Gamma}(2,\beta) \\ W \sim U(0,1) \end{array}}$$

2. Consider $U_1, \ldots, U_{300} \sim$ i.i.d. uniform$(-1, 1)$. Use the central limit theorem to find an approximation of the distribution of $\sum_{i=1}^{300} U_i/10$.

$\bar{U} = \frac{1}{300} \sum U_i$, $\mu = EU_1 = 0$, $\sigma^2 = Var(U_1) = \frac{2^2}{12} = \frac{1}{3}$

By CLT, $\frac{\sqrt{n}(\bar{U} - \mu)}{\sigma} \sim N(0,1)$ approximately

$$\frac{\sqrt{n}(\bar{U}-\mu)}{\sigma} = \frac{\sum U_i}{\sqrt{n}\sigma} = \frac{\sum U_i}{10}$$

$$\boxed{\frac{1}{10} \sum_{i=1}^{300} U_i \text{ follows approximately } N(0,1)}$$

3. Suppose that $X|Y = p$ has a Binomial(n, p) distribution for $0 < p < 1$ and that Y has a U$(0, 1)$ distribution. Find $E(X)$ and $var(X)$.

$E[X|Y=p] = np$, $E(X|Y) = nY$, $Var(X|Y) = nY(1-Y)$

$\boxed{E(X) = E[E(X|Y)] = E(nY) = \frac{n}{2}}$

$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$$

$$= E(nY(1-Y)) + Var(nY)$$

$$= nE(Y) - nE(Y^2) + n^2 Var(Y)$$

$$= \frac{n}{2} - \frac{n}{3} + \frac{n^2}{12}$$

$$\boxed{= \frac{n}{6} + \frac{n^2}{12}}$$

4. Let $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim}$ expo(1) (with pdf $f(x) = e^{-x}1_{(0,\infty)}(x)$). For every $n \geq 1$, Define $Z_n = \max\{X_1, \ldots, X_n\}/\log n$.
(a) Find the pdf of Z_n and $E(Z_n)$.

$$t>0 \cdot \quad P(Z_n \le t) = P(\max\{X_1, \ldots, X_n\} \le t\log n)$$
$$= \{P(X_1 \le t\log n)\}^n \quad \text{since } P(X_1 \le x) = 1-e^{-x},\ x>0$$
$$= (1-n^{-t})^n$$

$$\boxed{f_{Z_n}(t) = n(1-n^{-t})\log n\,(n^{-t}),\quad t\in(0,\infty)}$$

$$E(Z_n) = \int_0^\infty t n^{-t}(1-n^{-t})\,dt = \int_0^\infty t e^{-t\log n}\,dt - \int_0^\infty t e^{-2t\log n}\,dt$$

$$\left(\text{Use } \lambda\int_0^\infty e^{-\lambda x} x\,dx = \frac{1}{\lambda}\right) \qquad = \frac{1}{(\log n)^2} - \frac{1}{4(\log n)^2} = \frac{3}{4(\log n)^2}$$

(b) Find $\lim_{n\to\infty} P(Z_n \le 1)$. Does $Z_n \xrightarrow{d} 0$?

$$P(Z_n \le 1) = \left(1-\frac{1}{n}\right)^n \longrightarrow \boxed{e^{-1}} \text{ as } n\to\infty$$

$\boxed{Z_n \not\xrightarrow{d} 0}$ [otherwise $P(Z_n \le 1)$ would have converged to 1 as $n \to \infty$]

(used convergence in prob $\Leftrightarrow$ convergence in distribution for degenerate limits.)

5. (X, Y) has a joint pdf

$$f(x,y) = \begin{cases} Cxy & \text{if } 0 < y < 1; 0 < x < y \\ 0 & \text{o.w.} \end{cases}$$

$$\int_{y=0}^{1} \int_{x=0}^{y} C(xy)\, dx\, dy = 1$$

$$\Rightarrow C \int_0^1 \frac{y^3}{2}\, dy = 1$$

$$\Rightarrow C = 8$$

(a) Find the conditional pdf of $X \mid Y = y$ for $0 < y < 1$.

$$\boxed{f_Y(y) = 4y^3 \, \mathbb{1}_{(0,1)}(y)}$$

$$f_{X|Y}(x|y) = \begin{cases} \frac{8xy}{y^3} = \frac{2x}{y^2} & \text{if } 0 < x < y \\ 0 & \text{o.w.} \end{cases}$$

(b) Find the conditional pdf of $Y \mid X = x$ for $0 < x < 1$.

$$f_X(x) = 4x(1 - x^2) \, \mathbb{1}_{(0,1)}(x)$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^2} & \text{if } x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

(c) Find $E(X \mid Y = y)$ for $0 < y < 1$.

$$\int x \, f_{X|Y}(x|y)\, dy$$

$$= \int_0^y \frac{x \cdot 2x}{y^2}\, dx = \frac{2y}{3}$$

$$\boxed{E(X \mid Y = y) = \frac{2y}{3}}$$

6. Let X, Y be independent random variables distributed as geometric(p). Set $Z = X + Y$. Find the conditional distribution of $X \mid Z = z$. Verify that $E(X) = E(E(X \mid Z))$.

$X+Y$ takes values in $\{2, 3, \ldots\}$

$$k \geq 2, \quad P(X+Y=k) = \sum_{j=1}^{\infty} P(X=j)\,P(Y=k-j) = \sum_{j=1}^{k-1} p(1-p)^{j-1}\,p(1-p)^{k-j-1}$$

$$= (k-1)p^2(1-p)^{k-2}, \quad k = 2, 3,$$

Fix $z \geq 2$

$P(X=x \mid Z=z) = 0$ if $x \geq z$

For $x = 1, \ldots, z-1$

$$P(X=x \mid Z=z) = \frac{P(X=x, Y=z-x)}{P(Z=z)} = \frac{P(X=x)\,P(Y=z-x)}{P(Z=z)} = \frac{1}{z-1}$$

$$E(X \mid Z=z) = \frac{1}{z-1}\sum_{x=1}^{z-1} x = \frac{z}{2} \qquad E(X \mid Z) = Z/2$$

$$\boxed{E(E(X \mid Z)) = \frac{EZ}{2} = \frac{EX + EY}{2} = EX} \quad (\text{as } EX = EY)$$

7. Let $(X, Y) \sim N_2(0, 0, 1, 1, \rho)$. Define $V = (X - \rho Y)/\sqrt{1-\rho^2}$, $W = Y$. Find the joint distribution of V and W.

(V, W) is a linear transformation of (X, Y)

So (V, W) has BVN distribution.

$W \sim N(0,1)$, $EV = 0$,

$$\mathrm{Var}(V) = \frac{1}{1-\rho^2}\left[\mathrm{var}(X) + \mathrm{Var}(Y)\rho^2 - 2\rho\,\mathrm{Cov}(X,Y)\right]$$

$$= \frac{1}{1-\rho^2}\left[1 + \rho^2 - 2\rho^2\right] = 1$$

$\therefore V \sim N(0,1)$

$$\mathrm{Cov}(V, W) = \frac{1}{\sqrt{1-\rho^2}}\left[\underbrace{\mathrm{Cov}(X,Y)}_{=\rho} - \rho\,\underbrace{\mathrm{Cov}(Y,Y)}_{=\mathrm{var}(Y)}\right]$$

$$= 0$$

$\boxed{(V, W) \sim N(0, 0, 1, 1, 0) \quad [\text{iid } N(0,1)]}$

8. Let X_n be a sequence of random variables with $X_n \sim U(1-1/n, 1)$. Show that $X_n \xrightarrow{P} 1$.

~~fix $\epsilon > 0$, find n s.t. $\epsilon > \frac{1}{n} \Rightarrow -\frac{1}{n} < -\epsilon$~~

$\cancel{X_n - 1 \sim U(-\tfrac{1}{n}, 0)}$

$$\cancel{P(|X_n - 1| > \epsilon)}$$

$$\cancel{= n \int_{-1/n}^{-\epsilon} dx}$$

$$\cancel{= n\left(\tfrac{1}{n} - \epsilon\right)}$$

$$X_n - 1 \sim U\left(-\frac{1}{n}, 0\right)$$

$$|X_n - 1| \sim U\left(0, \frac{1}{n}\right)$$

For any $\epsilon > 0$ $\quad P(|X_n - 1| > \epsilon) \leq \dfrac{E(|X_n - 1|)}{\epsilon} = \dfrac{1}{2n\epsilon} \longrightarrow 0$

as $n \to \infty$

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9. Let $X_1, \ldots, X_{10}$ be i.i.d. $N(0,1)$. Find $E[(X_1 - X_2 + X_3 - X_4 + \ldots + X_9 - X_{10})^2]$.

$$(X_1 - X_2 + X_3 - X_4 + \cdots + X_9 - X_{10})^2$$

$$= \sum_{i=1}^{10} X_i^2 + 2\underbrace{\left(-X_1X_2 + X_1X_3 + \cdots\right)}$$

terms of the form

$\pm X_i X_j, \; i \neq j$

$$E X_i X_j = 0 \quad \forall\, i \neq j$$

$$\Rightarrow E\left[(X_1 - X_2 + X_3 - X_4 + \cdots + X_9 - X_{10})\right]^2$$

$$= E \sum^{10} X_i^2 = \boxed{10}$$