THIS TEST IS PRINTED ON 12 PAGES.

Name: ..................................

**5326 Review** (Debdeep Pati) **FSUID:** ..................................

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\sum$ |
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For each problem a maximum of **12** points can be earned. You get 4 points for writing the exam.

**Work your problems in the space provided. Show all work clearly. Justify your answers. Draw a box around your $\boxed{\text{final answer}}$.**

$1_A(x) = 1$ **if** $x \in A$ **and** $0$ **otherwise.** log **denotes the natural logarithm with respect to base** $e$.

**1.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. U(0, 1) random variables. For any positive integer $n$, define $V_n = n\min\{X_1, \ldots, X_n\}$, where min stands for minimum. [U(0, 1) has p.d.f. $1_{(0,1)}(x)$]

**(a)** Find $\lim_{n\to\infty} P(V_n \geq 1)$.

$X_1, X_2, \cdots, X_n \overset{iid}{\sim} U(0,1)$

For any $n \in \mathbb{N}$, define $V_n = n \min\{X_1, \cdots, X_n\}$.

Since $X_{(1)} = \min\{X_1, \cdots, X_n\}$, so $V_n = n \cdot X_{(1)}$

$$\lim_{n\to\infty} P(V_n \geq 1) = \lim_{n\to\infty} P(n \cdot X_{(1)} \geq 1) = \lim_{n\to\infty} P\left(X_{(1)} \geq \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$$

$X_1, \cdots, X_n \overset{iid}{\sim} U(0,1)$

$Y = \min\{X_1, \cdots, X_n\}$

$P(Y \geq y) = (1-y)^n$

$f_Y(y) = n(1-y)^{n-1}$

**(b)** Write down the p.d.f. of $V_n$.

For $x \in (0, n)$,

We know $P(V_n \leq x) = 1 - P\left(X_{(1)} \geq \frac{x}{n}\right)$

$$= 1 - \left(1 - \frac{x}{n}\right)^n.$$

Thus

$$f_{V_n}(x) = n\left(1 - \frac{x}{n}\right)^{n-1} \quad (0 \leq x \leq n)$$

**2.** Let $(X,Y) \sim N_2(0,0,1,1,\rho)$ for some $\rho \in (-1,1)$ with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[x^2+y^2-2\rho xy\right]}, \quad x,y \in \mathbb{R}.$$

Consider the polar transform from $(X,Y)$ to $(R,\Theta)$ so that $X = R\cos\Theta, Y = R\sin\Theta$ with $R > 0$ and $\Theta \in (0, 2\pi)$.

**(a)** Find the joint p.d.f. of $R$ and $\Theta$.

$X = R\cos\theta$
$Y = R\sin\theta$

$$|J| = \begin{vmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial R} & \frac{\partial Y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix} = R$$

$$x^2+y^2-2\rho xy = R^2 - 2\rho R^2\cos\theta\sin\theta = R^2 - \rho R^2 \sin 2\theta = R^2[1-\rho\sin 2\theta]$$

$$f_{R,\theta}(R,\theta) = \frac{R}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}R^2[1-\rho\sin 2\theta]} \quad \left(R>0,\ \theta \in (0,2\pi)\right)$$

Note: $f_{R,\theta}(R,\theta) = f_{X,Y}(\omega_1(R,\theta), \omega_2(R,\theta)) \cdot |J|$

$\omega_1(R,\theta) = R\cos\theta$
$\omega_2(R,\theta) = R\sin\theta$
$|J| = R$

**(b)** Find the marginal p.d.f. of $\Theta$ **when** $\rho = 0$.

$$f_\theta(\theta) = \int_R f_{R,\theta}(R,\theta)\,dR = \int_0^\infty \frac{R}{2\pi\sqrt{1-\rho^2}} e^{-\frac{(1-\rho\sin 2\theta)}{2(1-\rho^2)}R^2}\,dR$$

$$= \int_0^\infty \frac{R}{2\pi} e^{-\frac{1}{2}R^2}\,dR \quad (\text{when } \rho = 0)$$

$$= \frac{1}{2\pi}\int_0^\infty R e^{-\frac{1}{2}R^2}\,dR$$

$t = \frac{1}{2}R^2$
$dt = R\,dR$

$$= \frac{1}{2\pi}\int_0^\infty e^{-t}\,dt = \frac{1}{2\pi}\left[-e^{-t}\right]_0^\infty = \frac{1}{2\pi}[1] = \frac{1}{2\pi}$$

**3.** A random trial can result in three possible outcomes: success (S), failure (F) and indeterminate (I), with probabilities $P(\{S\}) = P(\{F\}) = p$ and $P(\{I\}) = 1 - 2p$, where $0 < p < 1/2$. Based on a single trial, define random variables $W$ and $Z$ as follows:

$$W = \begin{cases} 1 & \text{if outcome is S or I} \\ 0 & \text{otherwise.} \end{cases} \qquad Z = \begin{cases} 1 & \text{if outcome is F or I} \\ 0 & \text{otherwise.} \end{cases}$$

Find $\text{cov}(W, Z)$.

For $0<p<1/2$,

$P(\{S\}) = P(\{F\}) = p$

$P(\{I\}) = 1-2p$

$\text{Cov}(W,Z) = E(WZ) - E(W)E(Z) = 1-2p-(1-p)^2 = -p^2$

where

$E(W) = P(W=1) = P(\{S\}) + P(\{I\}) = p + 1-2p = 1-p$

$E(Z) = P(Z=1) = P(\{F\}) + P(\{I\}) = p + 1-2p = 1-p$

$E(WZ) = P(W=1, Z=1) = P((\{S\} \cup \{I\}) \cap (\{F\} \cup \{I\}))$

$= P(\{I\}) = 1-2p$

$\lambda = \frac{1}{n}$



**4. (a)** Let $X_n$ be a sequence of random variables with $X_n \sim \text{Expo}(1/n)$. Show that $X_n \xrightarrow{P} 0$. [Expo($\beta$) p.d.f. is $\beta^{-1}e^{-x/\beta}1_{(0,\infty)}(x)$]

Let $X_n$ be a sequence of r.v's with $X_n \sim \exp(\frac{1}{n}) = n e^{-nx} I_{(0,\infty)}(x)$.

We want to show $X_n \xrightarrow{P} 0$.

For every $\epsilon > 0$,

$$P(|X_n - 0| > \epsilon) \le \frac{E(|X_n|)}{\epsilon} = \frac{1}{n\epsilon} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $X_n \xrightarrow{P} 0$.

**(b)** Let $X_n$ be a sequence of random variables with $E(X_n) = \mu$ and $\text{var}(X_n) = b_n$, where $b_n$ is a sequence satisfying $\lim_{n\to\infty} b_n = 0$. Show that $X_n \xrightarrow{P} \mu$.

For every $\epsilon > 0$,

$$P(|X_n - \mu| > \epsilon) \le P(|X_n - \mu|^2 > \epsilon^2)$$

$$= \frac{E|X_n - \mu|^2}{\epsilon^2} \quad \text{(Using Chebyshev's Inequality)}$$

$$= \frac{Var(X_n) + [E(X_n - \mu)]^2}{\epsilon^2}$$

$$= \frac{1}{\epsilon^2}\left(\frac{1}{n^2} + \frac{1}{n^2}\right) = \frac{2}{n^2\epsilon^2} \longrightarrow 0 \quad \text{as } n \to \infty$$

Thus, $X_n \xrightarrow{P} \mu$.

**5.** Let $X$ and $Y$ be independent Expo($\beta$) random variables. [Expo($\beta$) p.d.f. is $\beta^{-1}e^{-x/\beta}1_{(0,\infty)}(x)$] Consider the transformation $U = X+Y$, $V = Y$. Find the joint p.d.f. of $U$ and $V$.

$X \perp\!\!\!\perp Y$

$X, Y \sim \exp(\beta) = \frac{1}{\beta} e^{-x/\beta} \qquad ( \quad x \in (0, \infty) \; )$

$U = X+Y \qquad X = U - V$

$V = Y \qquad Y = V$

$$|J| = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \qquad (\because X \perp\!\!\!\perp Y)$$

$$= \frac{1}{\beta} e^{-x/\beta} \cdot \frac{1}{\beta} e^{-y/\beta}$$

$$= \frac{1}{\beta^2} e^{-(x+y)/\beta}$$

$x \in (0,\infty)$ and $y \in (0,\infty)$ $\Rightarrow$ $v > 0$, $\infty > u - v > 0$, $\infty > u > v > 0$

$$f_{U,V}(u,v) = f_{X,Y}(u-v, v) \cdot |J|$$

$$= \frac{1}{\beta^2} e^{-u/\beta} \qquad (0 < v < u < \infty)$$

**6.** $(X, Y)$ has a joint p.d.f.

$$f(x,y) = \begin{cases} 15x^2y & \text{if } 0<y<1; 0<x<y \\ 0 & \text{o.w.} \end{cases}$$

**(a)** Find the conditional p.d.f. of $X \mid Y = y$ for $0 < y < 1$.

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{15x^2y}{5y^4} = \frac{3x^2}{y^3} \quad (0<x<y<1)$$

$$f_{X,Y}(x,y) = 15x^2y \quad (0<y<1,\ 0<x<y)$$

$$f_Y(y) = \int_0^y 15x^2y\,dx = 15y\left[\frac{1}{3}x^3\right]_0^y = 5y^4 \quad (0<y<1)$$

**(b)** Find the conditional p.d.f. of $Y \mid X = x$ for $0 < x < 1$.

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2\cdot \cancel{15x^2}y}{\cancel{15x^2}(1-x^2)} = \frac{2y}{1-x^2} \quad (0<x<y<1)$$

$$f_X(x) = 15x^2\int_x^1 y\,dy = 15x^2\left[\frac{1}{2}y^2\right]_x^1 = \frac{15x^2}{2}(1-x^2)$$

**7.** Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim}$ Poisson($\lambda$) and set $T_n = X_1 + \ldots + X_n$. [Poisson($\lambda$) p.m.f. is $e^{-\lambda}\lambda^x/x!, x = 0, 1, \ldots$]

**(a)** Find $E(T_n^2)$. [**You may use that for** $Y \sim$ Poisson($\lambda$), $E(Y) = \text{var}(Y) = \lambda$]

$$X_1, X_2, \cdots, X_n \overset{iid}{\sim} \text{Pois}(\lambda)$$

$$T_n = \sum_{i=1}^{n} X_i \sim \text{Pois}(n\lambda)$$

$$E(T_n^2) = V(T_n) + (E(T_n))^2 = n\lambda + (n\lambda)^2$$

$$= n^2\lambda^2 + n\lambda$$

**(b)** Find sequences $a_n, b_n$ so that $(T_n - a_n)/b_n$ is approximately $N(0, 1)$.

$$Z = \frac{T_n - E(T_n)}{\sqrt{V(T_n)}} \sim N(0,1)$$

To get $\frac{T_n - a_n}{b_n} \sim N(0,1)$, we know $a_n = E(T_n) = n\lambda$ and

$b_n = \sqrt{V(T_n)} = \sqrt{n\lambda}$.

**8.** For any positive real number $x$, let $\lfloor x \rfloor$ denote the largest non-negative integer less than or equal to $x$. For example, $\lfloor 0.3 \rfloor = 0$, $\lfloor 1.7 \rfloor = 1$, $\lfloor 2 \rfloor = 2$ etc.

Let $X \sim$ Expo(1) with p.d.f. $e^{-x}1_{(0,\infty)}(x)$. Find the distribution of $\lfloor X \rfloor$.

**Hint:** What is the support of $\lfloor X \rfloor$? Is it discrete or continuous?

For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest non-negative integer $\leq x$.

Let $X \sim \exp(1) = e^{-x}$

Hint: Support of $\lfloor X \rfloor = \{0, 1, \cdots\}$, which is discrete.

$P(\lfloor X \rfloor = k) = P(k \leq X < k+1)$ by the given definition.

$$= \int_k^{k+1} e^{-x}\,dx$$

$$= e^{-x}\Big|_k^{k+1} = e^{-(k+1)} - e^{-k} = e^{-k}(e^{-1}-1)$$