THIS TEST IS PRINTED ON 12 PAGES.

Name: ....

5326 Review (Debdeep Pati) **FSUID:** ....

1	2	3	4	5	6	7	8	$\sum$

For each problem a maximum of **12** points can be earned. You get 4 points for writing the exam.

Work your problems in the space provided. Show all work clearly. Justify your answers. Draw a box around your final answer.

$1_A(x) = 1$ **if** $x \in A$ **and** 0 **otherwise.** $\log$ **denotes the natural logarithm with respect to base** e**.**

1. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. U$(0,1)$ random variables. For any positive integer n, define $V_n = n \min\{X_1, \ldots, X_n\}$, where min stands for minimum. [U$(0,1)$ has p.d.f. $1_{(0,1)}(x)$]

(a) Find $\lim_{n\to\infty} P(V_n \geq 1)$.

(b) Write down the p.d.f. of V_n.

2. Let $(X, Y) \sim N_2(0, 0, 1, 1, \rho)$ for some $\rho \in (-1, 1)$ with joint p.d.f.

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[x^2+y^2-2\rho xy\right]}, \quad x, y \in \mathbb{R}.$$

Consider the polar transform from (X, Y) to (R, Θ) so that $X = R\cos\Theta, Y = R\sin\Theta$ with $R > 0$ and $\Theta \in (0, 2\pi)$.

(a) Find the joint p.d.f. of R and Θ.

(b) Find the marginal p.d.f. of Θ **when** $\rho = 0$.

3. A random trial can result in three possible outcomes: success (S), failure (F) and indeterminate (I), with probabilities $P(\{S\}) = P(\{F\}) = p$ and $P(\{I\}) = 1 - 2p$, where $0 < p < 1/2$. Based on a single trial, define random variables W and Z as follows:

$$W = \begin{cases} 1 & \text{if outcome is S or I} \\ 0 & \text{otherwise.} \end{cases} \qquad Z = \begin{cases} 1 & \text{if outcome is F or I} \\ 0 & \text{otherwise.} \end{cases}$$

Find $\text{cov}(W, Z)$.

4. (a) Let X_n be a sequence of random variables with $X_n \sim \text{Expo}(1/n)$. Show that $X_n \xrightarrow{P} 0$. [Expo(β) p.d.f. is $\beta^{-1}e^{-x/\beta}1_{(0,\infty)}(x)$]

(b) Let X_n be a sequence of random variables with $E(X_n) = \mu$ and $\text{var}(X_n) = b_n$, where b_n is a sequence satisfying $\lim_{n\to\infty} b_n = 0$. Show that $X_n \xrightarrow{P} \mu$.

5. Let X and Y be independent Expo(β) random variables. [Expo(β) p.d.f. is $\beta^{-1}e^{-x/\beta}1_{(0,\infty)}(x)$]
Consider the transformation $U = X + Y$, $V = Y$. Find the joint p.d.f. of U and V.

6. (X, Y) has a joint p.d.f.

$$f(x, y) = \begin{cases} 15x^2y & \text{if } 0 < y < 1; 0 < x < y \\ 0 & \text{o.w.} \end{cases}$$

(a) Find the conditional p.d.f. of $X \mid Y = y$ for $0 < y < 1$.

(b) Find the conditional p.d.f. of $Y \mid X = x$ for $0 < x < 1$.

7. Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim}$ Poisson(λ) and set $T_n = X_1 + \ldots + X_n$. [Poisson(λ) p.m.f. is $e^{-\lambda}\lambda^x/x!, x = 0, 1, \ldots$]

(a) Find $E(T_n^2)$. [**You may use that for** $Y \sim$ Poisson(λ), $E(Y) = \text{var}(Y) = \lambda$]

(b) Find sequences a_n, b_n so that $(T_n - a_n)/b_n$ is approximately $N(0, 1)$.

8. For any positive real number x, let $\lfloor x \rfloor$ denote the largest non-negative integer less than or equal to x. For example, $\lfloor 0.3 \rfloor = 0$, $\lfloor 1.7 \rfloor = 1$, $\lfloor 2 \rfloor = 2$ etc.

Let $X \sim \text{Expo}(1)$ with p.d.f. $e^{-x} 1_{(0,\infty)}(x)$. Find the distribution of $\lfloor X \rfloor$.

Hint: What is the support of $\lfloor X \rfloor$? Is it discrete or continuous?