HW1 Solution for STAB32T

#3.28 Show that each of the following families is an exponential family.

(a) normal family with either parameter μ or σ known.

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2\right)$$

i) μ known

$$\begin{cases} c(\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} I_{(0,\infty)}(\sigma^2) \\ h(x) = 1 \\ \omega_1(\sigma^2) = -\frac{1}{2\sigma^2} \\ t_1(x) = (x-\mu)^2 \end{cases}$$

(ii) σ^2 known,

$$\begin{cases} c(\mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \\ h(x) = \exp\left(-\frac{1}{2\sigma^2}x^2\right) \\ \omega_1(\mu) = \mu \\ t_1(x) = \frac{1}{\sigma^2}x \end{cases}$$

(b) gamma family with either parameter α or β known or both unknown

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} = \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{x}{\beta}} \exp((\alpha-1)\log x)$$

$$\boxed{\begin{array}{l} (0 \le x < \infty) \\ \alpha,\beta > 0 \end{array}}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp\left((\alpha-1)\log x - \frac{x}{\beta}\right)$$

i) α known

$c(\beta) = \frac{1}{\beta^\alpha}$, $h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} (x>0)$, $\omega_1(\beta) = \frac{1}{\beta}$, $t_1(x) = -x$

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(ii) β known,

$$C(\alpha) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}, \quad h(x) = e^{\frac{-x}{\beta}} \ (x>0), \quad w_1(\alpha) = \alpha-1, \quad t_1(x) = \log x$$

(iii) α, β unknown

$$C(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}, \quad h(x) = I_{(x>0)}, \quad w_1(\alpha) = \alpha-1, \ t_1(x) = \log x,$$
$$w_2(\beta) = \frac{-1}{\beta}, \ t_2(x) = x.$$

(c) beta family with either parameter α or β known or both unknown.

$$f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad \boxed{0 \le x \le 1 \quad \alpha,\beta>0}$$

$$= \frac{1}{B(\alpha,\beta)} \exp\big((\alpha-1)\log x\big)\exp\big((\beta-1)\log(1-x)\big)$$

$$= \frac{1}{B(\alpha,\beta)} \exp\Big((\alpha-1)\log x + (\beta-1)\log(1-x)\Big)$$

i) α known,

$$C(\beta) = \frac{1}{B(\alpha,\beta)}, \quad h(x) = x^{\alpha-1} I_{[0,1]}(x), \quad w_1(\beta) = \beta-1 \ ; \ t_1(x) = \log(1-x)$$

ii) β known,

$$C(\alpha) = \frac{1}{B(\alpha,\beta)}, \quad h(x) = (1-x)^{\beta-1} I_{[0,1]}(x), \quad w_1(\alpha) = \alpha-1, \quad t_1(x) = \log x$$

iii) α, β unknown

$$C(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}, \quad h(x) = I_{[0,1]}(x), \quad w_1(\alpha) = \alpha-1, \ t_1(x) = \log x,$$
$$w_2(\beta) = \beta-1, \ t_2(x) = \log(1-x).$$

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(d) Poisson family

$$P(X=x|\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-\lambda}e^{(x\log\lambda)}}{x!}$$

$c(\lambda) = e^{-\lambda}$, $h(x) = \frac{1}{x!} I_{\{0,1,\ldots\}}(x)$, $\omega_1(\lambda) = \log\lambda$, $t_1(x) = x$.

(e) negative binomial family with r known, $0<p<1$

$$P(X=x|r,p) = \binom{x-1}{r-1} p^{r}(1-p)^{x-r}, \quad x = 0,1,2,\cdots, \quad 0<p<1$$

$$= \binom{x-1}{r-1}\left(\frac{p}{1-p}\right)^{r} \exp(x\log(1-p))$$

$c(p) = \left(\frac{p}{1-p}\right)^{r}$, $h(x) = \binom{x-1}{r-1} I_{\{0,1,\cdots\}}(x)$, $\omega_1(p) = \log(1-p)$, $t_1(x) = x$.

#3.29 For each family in EX 3.28, describe the natural parameter space.

(a) For $N(\mu, \sigma^2)$,

$$f(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-\mu^2/2\sigma^2}}{\sigma}\right) e^{\left(-x^2/2\sigma^2 + x\mu/\sigma^2\right)}$$

Then the natural parameter is $(\lambda_1, \lambda_2) = \left(-\frac{1}{2\sigma^2}, \frac{\mu}{\sigma^2}\right)$ with the natural parameter space $\{(\lambda_1, \lambda_2) : \lambda_1 < 0, \ -\infty < \lambda_2 < \infty\}$.

(b) For $\text{Gamma}(\alpha, \beta)$,

$$f(x) = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right) e^{\left((\alpha-1)\log x - x/\beta\right)}$$

Then the natural parameter is $(\lambda_1, \lambda_2) = (\alpha - 1, -1/\beta)$ with the natural parameter space $\{(\lambda_1, \lambda_2) : \lambda_1 > -1, \ \lambda_2 < 0\}$.

(c) For $\text{Beta}(\alpha, \beta)$,

$$f(x) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\left(e^{(\alpha-1)\log x + (\beta-1)\log(1-x)}\right)$$

Then the natural parameter is $(\lambda_1, \lambda_2) = (\alpha - 1, \beta - 1)$ with the natural parameter space $\{(\lambda_1, \lambda_2) : \lambda_1 > -1, \ \lambda_2 > -1\}$.

(d) For $\text{Poisson}(\theta)$,

$$f(x) = \left(\frac{1}{x!}\right)\left(e^{-\theta}\right) e^{x \log \theta}$$

Then the natural parameter is $\lambda = \log\theta$ with the natural parameter space $\{\lambda : -\infty < \theta < \infty\}$

(e) For Negative Binomial (r, p), r known,

$$P(X = x) = \binom{r+x+1}{x} p^r e^{x \log(1-p)}$$

so the natural parameter is $\lambda = \log(1-p)$ with its space $\{\lambda : \lambda < 0\}$.

#3.30 Use the identities of Theorem 3.4.2 to

(a) Calculate the variance of a binomial random variable

$$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} (1-p)^n \exp\left(x \log\left(\frac{p}{1-p}\right)\right)$$

$h(x) = \begin{cases} \binom{n}{x} & x=0,1,2,\cdots,n \\ 0 & \text{o.w.} \end{cases}$ $\quad c(p) = (1-p)^n, \quad 0<p<1$

$w_1(p) = \log\left(\frac{p}{1-p}\right), \quad 0<p<1 \quad$ and $\quad t_1(x) = x$

By Theorem 3.4.2,

$$Var\left[\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(X)\right] = \underbrace{-\frac{\partial^2}{\partial \theta_j^2} \log c(\theta)}_{\text{par(b)}} \underbrace{- E\left[\sum_{i=1}^{k} \frac{\partial^2 w_i(\theta)}{\partial \theta_j^2} t_i(x)\right]}_{\text{part(a)}}$$

i) For the part(a),

By Quotient Rule

$$\left(\frac{p}{1-p}\right)' = \frac{(1-p)-p(-1)}{(1-p)^2} = \frac{1}{(1-p)^2}$$

$$-E\left[\frac{\partial^2}{\partial p^2} \log\left(\frac{p}{1-p}\right) x\right] = -E\left[\frac{\partial}{\partial p}\left(\frac{1-p}{p}\frac{1}{(1-p)^2}\right) x\right]$$

$$= -E\left[\frac{\partial}{\partial p}\left(\frac{1}{p-p^2}\right) x\right] = -E\left[\frac{(2p-1)}{p^2(1-p)^2} x\right] = \frac{n(1-2p)}{p(1-p)^2} \quad (\because EX=np)$$

ii) For the part(b)

$$-\frac{\partial^2}{\partial \theta_j^2} \log c(\theta) = -\frac{\partial^2}{\partial p^2}\left(n \log(1-p)\right) = -n\left(\frac{\partial}{\partial p}\left(\frac{-1}{1-p}\right)\right) = \frac{n}{(1-p)^2}$$

Therefore,

$$Var\left(\frac{\partial}{\partial p} \log\left(\frac{p}{1-p}\right) x\right) = \frac{n}{(1-p)^2} + \frac{n(1-2p)}{p(1-p)^2} = \frac{n[p+(1-2p)]}{p(1-p)^2} = \underbrace{\frac{n}{p(1-p)}}_{(0<p<1)}$$

3.30

(b) Calculate the mean and variance of a Poisson (λ) random variable

$$P(X=x|\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x=0,1,2,\cdots$$

$c(\lambda)=e^{-\lambda}$, $h(x)=\frac{1}{x!}I_{\{0,1,\cdots\}}(x)$, $\omega_1(\lambda)=\log\lambda$, $t_1(x)=x$.

I) Variance of a Poisson (λ)

Simily solve. Then we can decompose $Var\left[\sum_{i=1}^{k}\frac{\partial \omega_i(\theta)}{\theta_j^2}t_i(x)\right]$ into two parts.

(i) For the part (a),

$$-E\left[\frac{\partial^2}{\partial\lambda^2}(\log\lambda)x\right] = -E\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{\lambda}\right)x\right] = -E\left[-\frac{1}{\lambda^2}x\right] = \frac{1}{\lambda^2}E(x) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

(ii) For the part (b),

$$-\frac{\partial^2}{\partial\lambda^2}\log e^{-\lambda} = -\frac{\partial^2}{\partial\lambda^2}(-\lambda) = \frac{\partial}{\partial\lambda}1 = 0$$

Thus

$$Var\left[\frac{\partial}{\partial\lambda}(\log\lambda)x\right] = \frac{1}{\lambda} \qquad (\lambda>0)$$

II) mean of a Poisson (λ)

By Theorem 3.4.2,

$$E\left[\sum_{i=1}^{k}\frac{\partial\omega_i(\theta)}{\partial\theta_j}t_i(X)\right] = -\frac{\partial}{\partial\theta_j}(\log c(\theta)) = -\frac{\partial}{\partial\lambda}(\log e^{-\lambda}) = \frac{\partial}{\partial\lambda}\lambda = 1$$

#3.32

(a) Let $w_i(\theta) = \theta_i \quad \forall i$

$$\frac{\partial \eta_i}{\partial \theta_j} = \delta_{ij} \quad \text{(Dirac delta function)} \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$W(\boldsymbol{\theta}) = (w_1(\boldsymbol{\theta}), W_2(\boldsymbol{\theta}), \cdots, W_K(\boldsymbol{\theta}))$$

vector

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_K)$$

$$\Leftrightarrow \boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_K) \text{ natural parameter.}$$

① $$\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x) = \sum_{i=1}^{k} \frac{\partial \theta_i}{\partial \theta_j} t_i(X) = \sum_{i=1}^{k} \delta_{ij} t_i(x) = 1 \cdot t_j(x)$$

② $$\sum_{i=1}^{k} \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X) = \frac{\partial}{\partial \theta_j}\left(1 \cdot t_j(x)\right) = 0$$

Thus, the identities of Theorem 3.4.2 simplify to

$$E(t_j(x)) = -\frac{\partial}{\partial \eta_j} \log c^*(\eta)$$

$$Var(t_j(x)) = -\frac{\partial^2}{\partial \eta_j^2} \log c^*(\eta)$$

#3.32

(b) Gamma pdf

$$f(x|a,b) = \frac{1}{\Gamma(a)\, b^a} \exp.\underbrace{\left(\underset{\theta_1}{(a-1)} \log x - \underset{\theta_2}{\frac{1}{b}} x \right)}_{\exp(\theta_1 \log x + \theta_2 x)}, \quad \begin{matrix} x > 0 \\ a, b > 0 \end{matrix}$$

$$C(\theta) = \frac{1}{\Gamma(a) b^a}, \ \theta = (a, b) \qquad \begin{cases} t_1(X) = \log x \\ t_2(X) = x \end{cases}$$

Let $\theta_1 = a - 1 \Rightarrow a = \theta_1 + 1 \quad (\theta_1 > -1)$

$\theta_2 = -\frac{1}{b} \Rightarrow b = -\frac{1}{\theta_2} \quad (\theta_2 < 0)$

$$C^*(\theta) = \frac{1}{\Gamma(\theta_1 + 1) \left(-\frac{1}{\theta_2}\right)^{\theta_1 + 1}}, \quad \theta = (\theta_1, \theta_2)$$

$$E(t_2(x)) = E(X) = -\frac{\partial}{\partial \theta_2} \log C^*(\theta) = -\frac{\partial}{\partial \theta_2}\left[-\log \Gamma(\theta_1 + 1) - (\theta_1 + 1) \log(-1/\theta_2) \right]$$

$$= \frac{\partial}{\partial \theta_2}\left[\Gamma(\theta_1 + 1) - (\theta_1 + 1) \log(-\theta_2) \right]$$

$$= (\theta_1 + 1) \frac{1}{-\theta_2}$$

$$= (a - 1 + 1) \frac{1}{-(-\frac{1}{b})} = ab$$

$$Var(t_2(x)) = -\frac{\partial^2}{\partial \theta_2^2} \log C^*(\theta) = \frac{\partial}{\partial \theta_2}\left[(\theta_1 + 1)\left(\frac{-1}{\theta_2}\right) \right] = (\theta_1 + 1)\left(\frac{1}{\theta_2^2}\right)$$

$$= (a - 1 + 1)(-b)^2 = ab^2$$

#3.37 Given a pdf is symmetric about 0. Then μ is the median of the location-scale pdf $\frac{1}{\sigma}f\left(\frac{(x-\mu)}{\sigma}\right)$, $-\infty<x<\infty$

We know that the pdf is symmetric about 0. $\Rightarrow$ $f_X(c) = f_X(-c)$ for any c.

For $\epsilon>0$,

$$\frac{1}{\sigma}f\left(\frac{(\mu+\varepsilon)-\mu}{\sigma}\right) = \frac{1}{\sigma}f\left(\frac{\varepsilon}{\sigma}\right) \underset{\uparrow}{=} \frac{1}{\sigma}f\left(\frac{-\varepsilon}{\sigma}\right)$$

because f is symmetric about 0

$$= \frac{1}{\sigma}f\left(\frac{(\mu-\varepsilon)-\mu}{\sigma}\right)$$

Thus, the location-scale pdf is symmetric about μ.

But from Ex2.26(b), we know that if $f(x)$ is symmetric about μ for a constant μ, then μ is the median of X.

Hence, μ is the median of the location-scale pdf.

#3.38. Let Z a r.v. with pdf $f(z)$.

Define $\alpha = P(Z>z_\alpha) = \int_{z_\alpha}^{\infty} f(z)\,dz$

Given $\begin{cases} x_\alpha = \sigma z_\alpha + \mu \\ f(x) = \frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right) \end{cases}$

We want to show $P(X>x_\alpha)=\alpha$.

we know

$\boxed{Z=\frac{X-\mu}{\sigma} \Rightarrow X=\sigma Z+\mu}$

$$P(X>x_\alpha) = P(\sigma Z+\cancel{\mu} > \sigma z_\alpha+\cancel{\mu})$$
$$= P(Z>z_\alpha) = \alpha \text{ by Theorem 3.5.6.}$$

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#3.39 location-scale Cauchy pdf

$$f(x|\mu,\sigma) = \frac{1}{\sigma\pi\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)}, \quad -\infty<x<\infty$$

(a) μ is the median of the distribution of X, that is,

$$P(X \ge \mu) = P(X \le \mu) = \frac{1}{2}.$$

We know that if the pdf is symmetric about 0, then 0 must be the median from Ex3.37

Set $\mu=0$ and $\sigma=1$.

$$P(X\ge 0) = P(Z \ge 0) = \int_0^\infty \frac{1}{\pi}\frac{1}{(1+z^2)}\,dz$$

$$(\because Z = \frac{X-\mu}{\sigma} = X)$$

$$= \frac{1}{\pi}\tan^{-1}(z)\Big|_0^\infty = \frac{1}{\pi}\left(\frac{\pi}{2}-0\right) = \frac{1}{2}.$$

Since $P(X\ge\mu)=\frac{1}{2}$ for μ, μ is the median of $f(x)$.

(b) $\mu+\sigma$ and $\mu-\sigma$ are the quantiles of the distribution of X, that is,

$$P(X\ge \mu+\sigma) = P(X\le \mu-\sigma) = \frac{1}{4}.$$

Set $\mu=0$ and $\sigma=1$. Then we have

$$P(Z\ge 1) = \frac{1}{\pi}\tan^{-1}(z)\Big|_1^\infty = \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{1}{4}.$$

By the part (a), we know the pdf is symmetric about 0, so

$$P(Z\ge 0) = P\left(\frac{X-\mu}{\sigma}\ge 0\right) = P(X\ge\mu) = \frac{1}{2}$$ and we just obtained that

$$P(Z\ge 1) = P\left(\frac{X-\mu}{\sigma}\ge 1\right) = P(X\ge \mu+\sigma) = \frac{1}{4}.$$

By symmetry, $\frac{1}{4} = P(X\ge\mu+\sigma) = P(Z\ge 1) = P(Z\le -1) = P(X\le \mu-\sigma) = \frac{1}{4}$.

Thus, $\mu+\sigma$ and $\mu-\sigma$ are the quantiles of $f(x)$.

Chpt 6

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#6.1 Let $X \sim N(0, \sigma^2)$

By Factorization Theorem, $|X|$ is sufficient since $T(x) \sim N(0, \sigma^2)$.

Let $T(x) = |x|$. $f(x|\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}\sigma} e^{-|x|^2/2\sigma^2} = g(T(x)|\sigma^2) \cdot \underbrace{1}_{h(x)}$

#6.2 $f_{X_i}(x|\theta) = \begin{cases} e^{i\theta - x}, & x \ge i\theta \\ 0 & x < i\theta \end{cases}$

Let $T(x) = \min_i \left(\frac{X_i}{i}\right)$

The joint pdf $f(x_1, x_2, \cdots, x_n|\theta) = \prod_{i=1}^{n} e^{i\theta - x_i} I_{[i\theta, \infty)}(x_i)$ $(i > 0)$

$= \underbrace{e^{in\theta} I_{[i\theta, \infty)}(T(x))}_{g(T(x)|\theta)} \underbrace{e^{-\Sigma x_i}}_{h(x)}$ by factorization

Note that all $x_i > i\theta$ is satisfied iff $\min_i \left(\frac{x_i}{i}\right) > \theta$.

Thus, $T(X)$ is sufficient for θ

#6.3 Let $x_{(1)} = \min_i x_i$. Then we have the joint pdf

$$f(x_1, \ldots, x_n | \mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma} e^{-(x_i - \mu)/\sigma}, \quad \frac{\mu < x < \infty}{0 < \sigma < \infty}$$

$$= \frac{1}{\sigma^n} \cdot e^{-\sum_{i=1}^{n}(x_i - \mu)/\sigma}$$

$$= \underbrace{\left(\frac{e^{\mu/\sigma}}{\sigma}\right)^n e^{-\sum_{i=1}^{n} x_i} I_{(\mu, \infty)}(x_{(1)})}_{g(x_{(1)}, \sum_{i=1}^{n} x_i | \mu, \sigma)} \cdot \underbrace{1}_{h(x)}$$

Thus, $(X_{(1)}, \sum_{i=1}^{n} x_i)$ is a sufficient statistic for (μ, σ) by the Factorization Theorem.

#6.4 Prove Theorem 6.2.10.

Let $X_1, \dots, X_n \overset{iid}{\sim}$ a pdf or pmf exponential family given by

$$f(x|\theta) = h(x)\, c(\theta) \exp\left[\sum_{i=1}^{k} w_i(\theta) t_i(x)\right], \text{ where } \theta = (\theta_1, \theta_2, \dots, \theta_d) \text{ for } d \le k.$$

Then the joint pdf of $f(x|\theta)$ is

$$\prod_{j=1}^{n} f(x_j|\theta) = \prod_{j=1}^{n} h(x_j)\, c(\theta) \exp\left\{\sum_{i=1}^{k} w_i(\theta) t_i(x_j)\right\}$$

$$= \underbrace{c(\theta)^n \exp\left[\sum_{i=1}^{k} w_i(\theta) \sum_{j=1}^{n} t_i(x_j)\right]}_{g(T(x)|\theta)} \underbrace{\prod_{j=1}^{n} h(x_j)}_{h(x)}$$

where $T(x) = \left(\sum_{i=1}^{k}\sum_{j=1}^{n} t_i(x_j)\right) = \left(\sum_{j=1}^{n} t_1(x_j), \sum_{j=1}^{n} t_2(x_j), \dots, \sum_{j=1}^{n} t_k(x_j)\right)$

Thus $T(x)$ is a sufficient statistic for θ by the Factorization Theorem.

#6.5

Let $X_1, \dots, X_n$ are independt r.v. with a pdf is given by

$$f(x_i|\theta) = \begin{cases} \frac{1}{2i\theta} & -i(\theta-1) < x_i < i(\theta+1) \\ 0 & \text{o.w.} \end{cases} \qquad (\theta > 0)$$

Then the joint pdf of $f(x_i|\theta)$ is

$$\prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{2i\theta} I_{(-i(\theta-1) < x_i < i(\theta+1))}$$

$$= \underbrace{\left(\frac{1}{2\theta}\right)^n \left(\prod_{i=1}^{n} \frac{1}{i}\right) I_{\left(\min_i \frac{x_i}{i} \ge -(\theta-1)\right)} I_{\left(\max_i \frac{x_i}{i} \le \theta+1\right)}}_{g\left(\min_i\left(\frac{x_i}{i}\right), \max_i\left(\frac{x_i}{i}\right) \middle| \theta\right)} \cdot \underbrace{1}_{h(x)}$$

$-i(\theta-1) < x_i < i(\theta+1)$

$-(\theta-1) < \frac{x_i}{i} < (\theta+1)$

$\uparrow$ $\min\left(\frac{x_i}{i}\right)$ $\quad \uparrow$ $\max\left(\frac{x_i}{i}\right)$

Thus, $\left(\min_i\left(\frac{x_i}{i}\right), \max_i\left(\frac{x_i}{i}\right)\right)$ is a sufficient statistic for θ.

#6.6 Let $X_1, \cdots, X_n \sim$ Gamma(α, β)

Gamma pdf is $f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$ $\quad (x \geq 0, \; \alpha, \beta > 0)$

Then the joint pdf of $f(x|\alpha,\beta)$ is

$$\prod_{i=1}^{n} f(x_i|\alpha,\beta) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-x_i/\beta}$$

$$= \underbrace{\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\sum_{i=1}^{n} x_i/\beta}}_{g\left(\prod_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i \mid \alpha, \beta\right)} \cdot \underbrace{1}_{h(x)}$$

Thus $\left(\prod_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i\right)$ is sufficient for (α, β).

#6.7 Assume x and y are independent.

continuous Uniform $f(x|\theta_1,\theta_3) = \frac{1}{\theta_3 - \theta_1}$ $\quad (\theta_1 \leq x \leq \theta_3)$

$f(y|\theta_2,\theta_4) = \frac{1}{\theta_4 - \theta_2}$ $\quad (\theta_2 \leq y \leq \theta_4)$

The bivariate pdf is

$$f(x,y|\theta_1,\theta_2,\theta_3,\theta_4) = \frac{1}{(\theta_3-\theta_1)} I_{(\theta_1,\theta_3)}(x) \frac{1}{(\theta_4-\theta_2)} I_{(\theta_2,\theta_4)}(y)$$

Thus, $(x_{(1)}, x_{(n)}, y_{(1)}$ and $y_{(n)})$ is sufficient statistic for $(\theta_1, \theta_2, \theta_3, \theta_4)$.

Then the joint pdf of $f(x,y|\theta_1,\theta_2,\theta_3,\theta_4)$ is

$$\prod_{i=1}^{n} \frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)} I_{(\theta_1,\theta_3)}(x_i) I_{(\theta_2,\theta_4)}(y_i)$$

$$= \underbrace{\left(\frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)}\right)^n I_{(\theta_1,\infty)}(x_{(1)}) I_{(-\infty,\theta_3)}(x_{(n)}) I_{(\theta_2,\infty)}(y_{(1)}) I_{(-\infty,\theta_4)}(y_{(n)})}_{g(x_{(1)}, x_{(n)}, y_{(1)}, y_{(n)} | \theta_1,\theta_2,\theta_3,\theta_4)} \cdot \underbrace{1}_{h(x)}$$

where $x_{(1)} = \min\{x\}_{i=1}^{n}$, $x_{(n)} = \max\{x_i\}_{i=1}^{n}$, $y_{(1)} = \min\{y\}_{i=1}^{n}$ and $y_{(n)} = \max\{y\}_{i=1}^{n}$.

STA5327: EX 3.36 for Homework 1

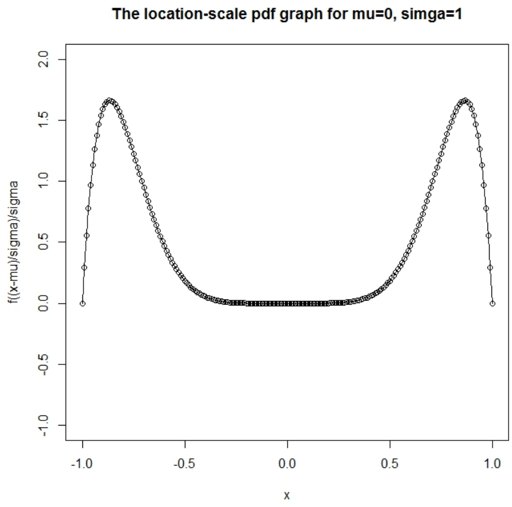

Figure 1: $\mu = 0, \sigma = 1$

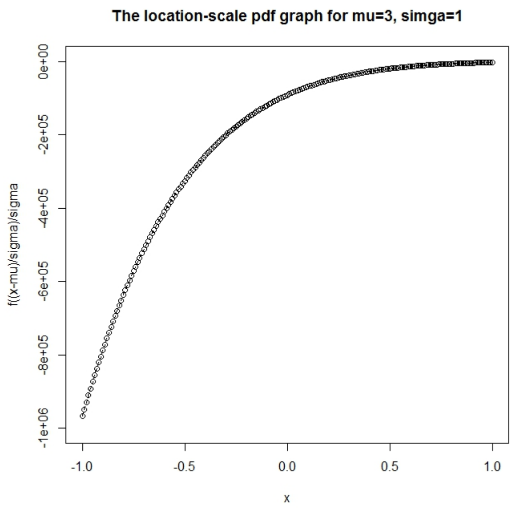

Figure 2: $\mu = 3, \sigma = 1$

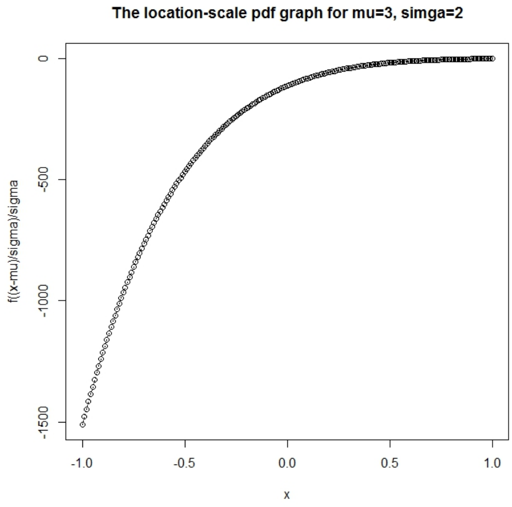

Figure 3: $\mu = 3, \sigma = 2$