Hw1 for 6448 (due Feb 7) (Debdeep Pati)

1. Suppose $(X_n, \mathcal{F}_n)$ is an adapted L_1 sequence and $E(X_{n+1} \mid \mathcal{F}_n) = a_n X_n + b_n$ for sequences $a_n, b_n \in \mathbb{R}$. How to convert it to a martingale?

2. $(X_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale. Let $\mathcal{F}_\infty = \sigma(\cup_{n\geq 0}\mathcal{F}_n)$ and $\sup E|X_n| = K$. ($E|X_n| \uparrow K$ since $|X_n|$ is a submartingale). Then the following are equivalent.

   (a) $K < \infty$, $X_n \overset{L_1}{\to} X_\infty$.

   (b) $(X_n, \mathcal{F}_n)_{0 \leq n \leq \infty}$ is a martingale.

   (c) $K < \infty$, $E|X_\infty| = K$.

   (d) $\{X_n\}_{n \geq 0}$ is uniformly integrable.

   Explain clearly all the steps in your proof.

3. Let $p > 1$. $\{X_n\}$ is L_p-bounded martingale or a non-negative sub-martingale. Then

   (a) $\{X_n\}$ is uniformly integrable.

   (b) $X_n \overset{L_p}{\to} X_\infty$.

   Show with a counter-example that one cannot get rid of the non-negativity in case of sub-martingale.

4. Simulate a branching process as follows: Start with 1 member, namely the population size is $Z_0 = 1$. Let $\xi_i^n, i = 1, 2, \ldots, n$ denote the number of children of ith individual in the nth generation. Assume $\xi_i^n \overset{iid}{\sim}$ Poisson(μ). Let Z_n denote the population size in the nth generation. It is shown in class that $X_n = Z_n/\mu^n$ defines a martingale. Plot X_n and Z_n with respect to $n = 1, 2, \ldots, 500$ in the following cases:

   (a) $\mu = 0.5$

   (b) $\mu = 1$

   (c) $\mu = 1.5$

   Explain the observations particularly the case $\mu = 1.5$. Attach your code with the homework.