HW 11 Solution.

7.1 $X = \begin{cases} 1 & \text{Head} \\ 0 & \text{Tail} \end{cases}$

Y: the value show up on the die $1, 2, \ldots, 6$

$P(i,j) = P(X=i, Y=j) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$

$E = \sum_{j=1}^{6} 2 \cdot j \cdot P(1,j) + \sum_{j=1}^{6} \frac{j}{2} \cdot P(0,j)$

$= (2+4+6+8+10+12) \times \frac{1}{12} + (\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\frac{4}{2}+\frac{5}{2}+\frac{6}{2}) \times \frac{1}{12}$

$= \frac{42+10.5}{12} = \boxed{\frac{52.5}{12}} = \boxed{4.375}$

7.2 a) $6 \times 6 \times 9 = \boxed{324}$

b) $X = (6-S)(6-W)(9-R)$

c) $EX = (6)(6)(6) P\{S=0, W=0, R=3\}$

$+ (6)(3)(9) P(0,3,0)$

$+ (3)(6)(9) P(3,0,0)$

$+ (6)(5)(7) P(0,1,2)$

$+ (5)(6)(7) P(1,0,2)$

$+ (6)(4)(8) P(0,2,1)$

$+ (4)(6)(8) P(2,0,1)$

$+ (5)(4)(9) P(1,2,0)$

$+ (4)(5)(9) P(2,1,0)$

$+ (5)(5)(8) P(1,1,1)$

$= \frac{1}{\binom{21}{3}} \times \left(216 \cdot \binom{9}{3} + 324 \binom{6}{3} + 420 \cdot 6 \cdot \binom{9}{2} + 384 \cdot \binom{6}{2} + 360 \binom{6}{2} \cdot 6 + 200 \times 6 \times 6 \times 9\right) = \boxed{198.\ldots}$

7.3 Let X : # of the gambling the person totally takes.

X	0	1	2	3	4	5	$\cdots$	n	$\cdots$
Prob	0	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}$	$(\frac{1}{2})^3$	$(\frac{1}{2})^4$	$(\frac{1}{2})^5$	$\cdots$	$(\frac{1}{2})^n$	$\cdots$
W	0	1	0	-1	-2	-3	$\cdots$	$-n+2$	$\cdots$

a) $P(W>0) = P(X=1) = \boxed{\frac{1}{2}}$

b) $P(W<0) = 1 - P(W>0) - P(W=0)$

$= 1 - \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2}$

$= \boxed{\frac{1}{4}}$

c) $EW = \frac{1}{2} + 0 - \frac{1}{2^3} - \frac{2}{2^4} - \frac{3}{2^5} - \cdots - \frac{n-2}{2^n} - \cdots$

Let $S = \frac{1}{2} - \frac{1}{2^3} - \frac{2}{2^4} - \frac{3}{2^5} - \cdots - \frac{k-2}{2^k}$

$\frac{1}{2}S = \frac{1}{4} \quad - \frac{1}{2^4} - \frac{2}{2^5} - \cdots - \frac{k-3}{2^k} - \frac{k-2}{2^{k+1}}$

$S - \frac{1}{2}S = \frac{1}{4} - \frac{1}{2^3} - \frac{1}{2^4} - \frac{1}{2^5} - \cdots - \frac{1}{2^k} + \frac{k-2}{2^{k+1}}$

$= \frac{1}{4} + \frac{k-2}{2^{k+1}} \cancel{+(\frac{1}{2})} - \left(\frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^k}\right)$

$= \frac{2^{k-1}+k-2}{2^{k+1}} \cdot \left[-\left(\frac{\frac{1}{2^3}\left[1-(\frac{1}{2})^{k-2}\right]}{1-\frac{1}{2}}\right)\right]$

$= \frac{2^{k-1}+k-2}{2^{k+1}} \cancel{\times}\left(-\left[(\frac{1}{2})^2\cdot\left(1-(\frac{1}{2})^{k-2}\right]\right)\right)$

$\to -\infty, \ k \to \infty.$

So $\frac{1}{2}S \to -\infty$, $S \to -\infty$

So $EW = -\infty$.

7.8 $E[\text{number of occupied tables}]$

$= E[\sum_{i=1}^{N} X_i]$

$= \sum_{i=1}^{N} E X_i$

Now, $E X_i = P\{i^{th}$ arrival is ~~not~~ not friends with any first $i-1\}$

$= (1-p)^{i-1}$

$E[\text{number of occupied tables}] = \sum_{i=1}^{N} (1-p)^{i-1} = \boxed{\frac{1-(1-p)^N}{p}}$

7.9. $X_j = \begin{cases} 1 & \text{urn } j \text{ is empty} \\ 0 & \text{otherwise.} \end{cases}$

$E X_j = P\{\text{ball } i \text{ is not in urn } j,\ i \geq j\} = \prod_{i=j}^{n} (1 - \frac{1}{i})$

a) $E[\text{number of empty urns}] =$

$= \sum_{j=1}^{n} \sum_{i=j}^{n} (1 - \frac{1}{i})$

b) $P\{\text{none are empty}\} = P\{\text{ball } j \text{ is in urn } j, \text{ for all } j\}$

$= \prod_{j=1}^{n} \frac{1}{j}$

7.10 $X_i = \begin{cases} 1 & \text{if trial } i \text{ is success} \\ 0 & \text{otherwise} \end{cases}$

a) $\boxed{0.6}$ This occurs when $P\{X_1 = X_2 = X_3 = 1\}$

It is the largest possible value because

$1.8 = \sum_{i=1}^{3} P\{X_i = 1\} = 3\, P\{X_i = 1\}$

$\Rightarrow P\{X_i = 1\} = 0.6$

So $P\{X=3\} = P\{X_1 = X_2 = X_3 = 1\}$

b) $\boxed{0}$ Let $X_1 = \begin{cases} 1 & \text{if } U \le 0.6 \\ 0 & \text{otherwise} \end{cases}$

$X_2 = \begin{cases} 1 & \text{if } U \le 0.4 \\ 0 & \text{otherwise} \end{cases}$

$X_3 = \begin{cases} 1 & \text{if } U \le 0.3 \\ 0 & \text{otherwise} \end{cases}$

It is not possible for all X_i to equal 1.

7.11 $X_i = \begin{cases} 1 & \text{if changeover occurs on } i^{th} \text{ flip} \\ 0 & \text{otherwise} \end{cases}$

$EX_i = P\{i-1 \text{ is } H,\ i \text{ is } T\} + P\{i-1 \text{ is } T,\ i \text{ is } H\}$

$= 2(1-p)\cdot p,\ i \geq 2.$

$E[\text{number of changeovers}] = E[\sum X_i] = \sum_{i=1}^{n} E[X_i] = 2(n-1)(1-p)$