Hw1 Solution

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Solution to Problem 3a: We will only provide the proof for L_p bounded martingale. The proof for L_p bounded nonnegative submartingale would be similar. Let $M := \sup_{n\geq 1} E|X_n|^p$. Fix $\epsilon > 0$. We will show that there exists K_0 such that for $K \geq K_0$,

$$\sup_{n\geq 1} E[|X_n| I_{|X_n|>K}] < \epsilon.$$

Notice that since $|X_n|^p/|X_n|$ is increasing in $|X_n|$ for $p > 1$, we can find a K_0 such that for $K \geq K_0$,

$$\{|X_n| > K\} \subset \{|X_n|^p/|X_n| > M/\epsilon\}.$$

Hence for $K \geq K_0$

$$\sup_{n\geq 1} \int_{|X_n|>K} |X_n| dP \leq \sup_{n\geq 1} \frac{\epsilon}{M} \int_{|X_n|>K} |X_n|^p dP \leq \epsilon.$$

Solution to Problem 3b: *There are many ways to prove this. The standard approach is to take the route to prove Theorem 5 in the class notes. Here is a tricky alternative route.*
Let $X_n^* = \max\{|X_1|, \ldots, |X_n|\}$. Then by Corollary 1 in the class notes,

$$\sup_{n\geq 1} \{E[(X_n^*)]^p\}^{1/p} \leq \sup_{n\geq 1} \frac{p}{p-1} \{E[|X_n|^p]\}^{1/p} < \infty.$$

Since $X_n^* \uparrow X_\infty^* = \sup_{n\geq 1} |X_n|$, MCT implies $E[(X_\infty^*)]^p\}^{1/p} < \infty$. Hence $\sup_{n\geq 1} |X_n|^p \in L_1$. Thus the random variables $\{|X_n|, n \geq 1\}^p$ are uniformly bounded by a random variable in L_1, so DCT implies the conclusion.