Hw2 for 6448 (due Feb 23) (Debdeep Pati)

Bahadur's representation of sample quantiles: Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with cdf $F(x) = P(X_i \leq x)$ and pdf $f(x)$. For $0 < p < 1$, let $\xi_p = \inf\{x : F(x) \geq p\}$ denote the p-th quantile of F. Denote the empirical distribution function by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{X_i \leq x}.$$

Denote by $\xi_{n,p}$ the pth quantile of F_n. Bahadur (1966) showed that if $f(\xi_p) > 0$,

$$\xi_{n,p} = \xi_p + \frac{p - F_n(\xi_p)}{f(\xi_p)} + Y_n$$

where Y_n is a random variable satisfying $|Y_n| = o(n^{-1/2})$ almost surely. Recall $a_n \sim o(b_n)$ iff $a_n/b_n \to 0$.

1. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from distribution $F(x)$ that has density $f(x)$. Let μ be the mean and η be the median of the distribution. Furthermore, let $\bar{X}$ be the sample mean and $\tilde{X}$ be the sample median. Assume F has a finite variance σ^2, $F''(x)$ exists at η and $f(\eta) > 0$.
   (a) Show that the asymptotic covariance between $\bar{X}$ and $\tilde{X}$ is $\frac{E(|X-\eta|)}{2nf(\eta)}$ and derive the asymptotic bivariate distribution for $(\bar{X}, \tilde{X})^{\mathrm{T}}$.
   (b) Assume $\mu \neq 0$ and find the asymptotic distribution for $V_n = \tilde{X}/\bar{X}$.
   (c) Assume $\eta = \mu$ and let $T_n = c\bar{X} + (1-c)\tilde{X}$. Find the value of c that minimizes the asympottic variance of T_n. Discuss how c could be estimated to get an estimator for μ.
2. Generate n observations from

$$0.5\,\mathrm{N}(0,1) + 0.5\exp(\text{mean} = 2).$$

   Then $EX_1 = \mu_0 = 1$. Test $H_0 : \mu = \mu_0$ vs. $H_1 : \mu > \mu_0$ at 5% using significance level using
   (a) Standard normal approximation.
   (b) Empirical likelihood.

   Try $n = 50, 200$. Find 0.05% confidence intervals of μ in all the cases using i) Bootstrap and ii) χ^2 approximation.
3. Find confidence intervals for $0.25, 0.5$ and 0.75-quantiles in the previous example when $n = 50, 200$. Compare with the asymptotic normal confidence intervals for quantiles (can be derived following a similar exercise as in Problem 1).