Hw2 - Problem 1 Solutions

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1(a) Since $F''(x)$ exists at η and $f(\eta) > 0$, we have by the Bahadur representation that

$$\tilde{X} - \eta = \frac{\frac{1}{2} - F_n(\eta)}{f(\eta)} + o_p(n^{-1/2}),$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \leq x\}}$ is the empirical distribution function. Thus, the asymptotic distribution of $\tilde{X} - \eta$ s the same as that of $\bar{Y} = \frac{\frac{1}{2} - F_n(\eta)}{f(\eta)}$, which is $N(0, 1/[n(2f(\eta))^2])$ by the CLT.

By the CLT, the asymptotic distribution of $(\bar{X}, \bar{Y})'$ is

$$\text{Normal}\{(\mu, 0)', [\sigma^2/n, \gamma/n; \gamma/n, 1/[n(2f(\eta))^2]]\},$$

where $\gamma = Cov(X_1, -\frac{1}{f(\eta) I_{\{x \leq \eta\}}})$. We can calculate γ as follows:

$$\begin{aligned}
\gamma =& -\frac{1}{f(\eta)} \int_{-\infty}^{\infty} x I_{\{x \leq \eta\}} f(x) dx + \frac{\mu}{2f(\eta)} \\
=& \frac{1}{f(\eta)} \int_{-\infty}^{\eta} x f(x) dx + \frac{\mu}{2f(\eta))} \\
=& \frac{1}{2f(\eta)} (\int_{\eta}^{\infty} x f(x) dx - \int_{-\infty}^{\eta} x f(x) dx) \\
=& \frac{1}{2f(\eta)} (\int_{\eta}^{\infty} (x - \eta) f(x) dx - \int_{-\infty}^{\eta} (x - \eta) f(x) dx) \\
=& \frac{E(|X - \eta|)}{2f(\eta)}.
\end{aligned}$$

Therefore, the asymptotic bivariate distribution of $(\bar{X}, \tilde{X})'$ is

$$\text{Normal}\{(\mu, \eta)', [\sigma^2/n, \gamma/n; \gamma/n, 1/[n(2f(\eta))^2]]\}.$$

1(b) Observe that

$$V_n = \frac{\eta}{\mu} + \frac{\tilde{X} - \eta}{\mu} - \frac{\eta}{\mu^2}(\bar{X} - \mu) + O_p(1/n).$$

Thus, by the CLT the asymptotic distribution of V_n is $N(\eta/\mu, \xi_n^2)$, where

$$\begin{aligned}
\xi_n^2 =& \frac{1}{n\mu^2}(\frac{1}{4f^2(\eta)} - \frac{\eta E(|X - \eta|)}{\mu f(\eta)} + \frac{\eta^2\sigma^2}{\mu^2}) \\
=& Var(\frac{\tilde{X} - \eta}{\mu} - \frac{\eta}{\mu^2}(\bar(X) - mu)) + o(1/n)
\end{aligned}$$

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Alternatively (and equivalently), we can apply the multivariate delta method, noting that

$$\frac{\partial(\eta/\mu)}{\partial\mu} = -\frac{\eta}{\mu}; \frac{\partial(\eta/\mu)}{\partial\eta} = \frac{1}{\mu}.$$

1(c) We have

$$\begin{aligned}
Var(T_n) =& \frac{c^2\sigma^2}{n} + \frac{2c(1-c)\gamma}{n} + \frac{(1-c)^2}{n[2f(\eta)]^2} + o(1/n) \\
=& \frac{c^2\sigma^2 + 2c(1-c)\gamma + (1-c)^2[2f(\eta)]^2}{n} + o(1/n).
\end{aligned}$$

The dominating term is minimized at

$$c = c_0 = \frac{1 - 4\gamma f^2(\eta)}{4\sigma^2 f^2(\eta) - 8\gamma f^2(\eta) + 1} = \frac{1 - 2E(|X-\eta|)f(\eta)}{4\sigma^2 f^2(\eta) - 4E(|X-\eta|)f(\eta) + 1}.$$

c_0 can be estimated with $\hat{c}_0$ obtained by estimating σ^2 with the sample variance, η with the sample median, $E(|X - \eta|)$ with the sample average absolute deviation of the observations from the sample median, and $f(\eta)$ with a nonparametric estimate of the density at η. Then the desired linear combination estimator for μ is $\hat{c}_0\bar{X} + (1 - \hat{c}_0)\tilde{X}$.