Homework 2 Solutions

September 12, 2013 STA 4442/5440

Solutions to the problems from Sheldon and Ross

10. Let $P(A)$ be the probability that a student wears a ring. Let $P(B)$ be the probability that a student wears a necklace. Then from the information given we have that

$$\begin{aligned} P(A) &= 0.2 \\ P(B) &= 0.3 \\ P((A^c \cap B^c)) &= 0.6. \end{aligned}$$

   a) By De Morgan's theorem, $P(A \cup B) = 1 - P((A \cup B)^c) = 1 - 0.6 = 0.4$.

   b) By the inclusion/exclusion identity for two sets

   $$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

   Hence

   $$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.2 + 0.3 - 0.4 = 0.1.$$

14. Following the hint given in the book, we let M denote the set of people who are married, W the set of people who are working professionals, and G the set of people who are college graduates. If we choose a random person and ask what the probability that he/she is either married or working or a graduate we are looking to compute $P(M \cup W \cup G)$. By the inclusion/exclusion theorem we have that the probability of this event is given by

$$\begin{aligned} P(M \cup W \cup G) = P(M) + P(W) + P(G) - P(M \cap W) - P(M \cap G) - P(W \cap G) \\ +P(M \cap W \cap G). \quad (1) \end{aligned}$$

Following the data given,

$$\begin{aligned} P(M) = 470/1000, P(G) = 525/1000, P(W) = 312/1000 \\ P(M \cap G) = 147/1000, P(M \cap W) = 86/1000, P(W \cap G) = 42/1000 \\ P(M \cap W \cap G) = 25. \end{aligned}$$

Using (1), we have

$$P(M \cap W \cap G) = 0.47 + 0.525 + 0.312 - 0.147 - 0.086 - 0.042 + 0.025 = 1.057 > 1$$

25. A sum of five has a probability of $P_5 = 1/9$ of occurring. A sum of seven has a probability of $P_7 = 1/6$ of occurring, so the probability that neither a five or a seven is given by $1 - 1/9 - 1/6 = 13/18$. Following the hint we let E_n be the event that a five occurs on the n-th roll and no five or seven occurs on the $(n-1)$ th rolls up to that point. Then

$$P(E_n) = \left(\frac{13}{18}\right)^{n-1}\frac{1}{9}.$$

Since we want the probability that a five comes first, this can happen at roll number one (n = 1), at roll number two (n = 2) or any subsequent roll. Thus the probability that a five comes first is given by

$$\begin{aligned}\frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{13}{18}\right)^{n-1} &= \sum_{n=0}^{\infty}\left(\frac{13}{18}\right)^{n} \\ &= \frac{1}{9}\frac{1}{1-13/18} = 2/5.\end{aligned}$$

45. a) If unsuccessful keys are removed as we try them, then the probability that the k-th attempt opens the door can be computed by recognizing that all attempts up to (but not including) the k-th have resulted in failures. Specifically, if we let N be the random variable denoting the attempt that opens the door we see that

$$\begin{aligned}P(N=1) &= \frac{1}{n} \\ P(N=2) &= \left(1-\frac{1}{n}\right)\frac{1}{n-1} \\ P(N=3) &= \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right)\frac{1}{n-2} \\ &\vdots \\ P(N=k) &= \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right)\cdots\left(1-\frac{1}{n-(k-2)}\right)\frac{1}{n-(k-1)}\end{aligned}$$

Hence

$$P(N=k) = \frac{1}{n}$$

b) If unsuccessful keys are not removed then the probability that the correct key is selected at draw k

$$PN = k = (1-p)^{k-1}p$$

Solutions to the theoretical exercises Sheldon and Ross

11. From the inclusion/exclusion identity for two sets we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Since $P(E \cup F) \leq 1$, the above becomes

$$\begin{aligned} & P(E) + P(F) - P(E \cap F) \leq 1 \\ \Rightarrow \quad & P(E \cap F) \geq P(E) + P(F) - 1 \end{aligned}$$

which is known as Bonferroni's inequality. Hence

$$P(E \cap F) \geq 0.9 + 0.8 - 1 = 0.7.$$

16. From Bonferronis inequality for two sets $P(E \cap F) \geq P(E) + P(F) - 1$, when we apply this identity recursively we see that

$$\begin{aligned} P(E_1 \cap E_2 \cap E_3 \cap \cdots E_n) &\geq P(E_1) + P(E_2 \cap E_3 \cap E_4 \cdots \cap E_n) - 1 \\ &\geq P(E_1) + P(E_2) + P(E_3 \cap E_4 \cap E_5 \cdots \cap E_n) - 2 \\ &\geq \cdots \\ &\geq P(E_1) + P(E_2) + \cdots + P(E_n) - (n-1). \end{aligned}$$

That the final term is n - 1 can be verified to be correct by evaluating this expression for n = 2 which yields the original Bonferroni inequality.

Solution to the "Socks in the drawer" problem

a) Total number of ways to choose is $\binom{2n}{2} = n(2n-1)$. For each color, there are $\binom{n}{2}$ pairs of the color. The total number of matching pairs is $2\binom{n}{2} = n^2 - n$. Hence the probability of choosing a matching pair of socks from a drawer with n white and n black socks is given by

$$\frac{n^2 - n}{n(2n-1)}$$

b) $\frac{n(n-1}{n(2n-1)} = \frac{(n-1)}{(2n-1)} \to 1/2$ as $n \to \infty$.

Solution to the "Boole's Inequality"

Observe that

$$\begin{aligned} & P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_2 \cap A_2) \\ \Rightarrow \quad & P(A_1 \cup A_2) \leq P(A_1) + P(A_2) \end{aligned}$$

When we apply this inequality recursively we see that

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup \cdots A_n) &\leq P(A_1) + P(A_2 \cup A_3 \cup \cdots A_n) \\ &\leq P(A_1) + P(A_2) + P(A_3 \cup A_4 \cup A_5 \cdots \cup A_n) \\ &\leq \cdots \\ &\leq P(A_1) + P(A_2) + \cdots + P(A_n). \end{aligned}$$