Homework 3 Solutions

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Solutions of problems from Sheldon Ross book (8th Edition) Chapter 3

11. Let B be the event that both cards are aces, A_s the event that the ace of spades is chosen and A the event that at least one ace is chosen.

    a) We are asked to compute $P(B|A_s)$. Using the definition of conditional probabilities we have that

    $$P(B \mid A_s) = \frac{P(B \cap A_s)}{P(A_s)}$$

    The event $B \cap A_s$ is the event that both cards are aces and one is the ace of spades. This event can be represented by the sample space

    $$\{(AD, AS), (AH, AS), (AC, AS)\}.$$

    where D, S, H, and C stand for diamonds, spades, hearts, and clubs respectively and the order of these elements in the set above does not matter. The event A_s is given by the set $\{AS, *\}$ where $*$ is a wild-card denoting any of the possible fifty-one other cards besides the ace of spades. Hence

    $$\begin{aligned} P(B \cap A_s) &= \frac{3}{\binom{52}{2}} \\ P(A_s) &= \frac{51}{\binom{52}{2}}. \end{aligned}$$

    Hence

    $$P(B \mid A_s) = 3/51 = 1/17.$$

    b) The event B has $\binom{4}{2}$ elements i.e. from the four total aces select two. Hence,

    $$P(B) = \frac{\binom{4}{2}}{\binom{52}{2}}$$

    The set A is the event that at least one ace is chosen. This is the complement of the set that no ace is chosen. No ace can be chosen in $\binom{48}{2}$ ways so that

    $$P(A) = \frac{\binom{52}{2} - \binom{48}{2}}{\binom{52}{2}}$$

Hence

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)} = \frac{\binom{4}{2}}{\binom{52}{2} - \binom{48}{2}} = 1/33.$$

12. We let E_i be the event that the ith actuarial exam is passed. Then the given probabilities can be expressed as

$$P(E_1) = 0.9, P(E_2 \mid E_1) = 0.8, P(E_3 \mid E_1, E_2) = 0.7.$$

a) The desired probability is given by $P(E_1 \cap E_2 \cap E_3)$ or conditioning we have

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1E_2) = 0.9 \cdot 0.8 \cdot 0.7 = 0.504.$$

b) The desired probability is given by $P(E_2^c \mid (E_1 \cap E_2 \cap E_3)^c)$ and can be expressed as

$$P(E_2^c \mid (E_1 \cap E_2 \cap E_3)^c) = \frac{P(E_2^c \cap (E_1 \cap E_2 \cap E_3)^c)}{P(E_1 \cap E_2 \cap E_3)^c}.$$

are the only ways that one can not pass all three tests i.e. one must fail one of the first three tests. Hence

$$(E_1 \cap E_2 \cap E_3)^c = E_1^c \cup (E_1 \cap E_2^c) \cup (E_1 \cap E_2 \cap E_3^c).$$

From the above set identity the event $E_2^c \cap (E_1 \cap E_2 \cap E_3)^c$ is composed of only one set, namely $E_1E_2^c$, since if we dont pass the second test we dont take the third test. We now need to evaluate the probability of this event. We find

$$P(E_1E_2^c) = (1 - P(E_2 \mid E_1))P(E_1) = (1 - 0.8)(0.9) = 0.18$$

With this the conditional probability sought is given by $0.18/(1-0.504) = 0.3629$

22. a) The probability that no two dice land on the same number means that each die must land on a unique number. To count the number of such possible combinations we see that there are six choices for the red die, five choices for the blue die, and then four choices for the yellow die yielding a total of $6 \ 5 \ 4 = 120$ choices where each die has a different number. There are a total of 6^3 total combinations of all possible die through giving a probability of $120/6^3 = 5/9$.

b) We are asked to compute $P(B < Y < R \mid E)$ where E is the event that no two dice land on the same number. From Part (a) above we know that the count of the number of rolls that satisfy event E is 120. Now the number of rolls that satisfy the event $B < Y < R$ can be counted as follows. For example, if R shows a roll of three then the only possible valid rolls where $B < Y < R$ for B and

Y are $B = 1$and $Y = 2$. If R shows a four then we have $\binom{3}{2}$ possible choices i.e. either $(B = 1, Y = 2), (B = 1, Y = 3), (B = 2, Y = 3)$ for the possible assignments to the two values for the B and Y die. If R = 5 we have $\binom{4}{2} = 6$ possible assignments to B and Y . Finally, if R = 6 we have $\binom{5}{2} = 10$ possible assignments to B and Y . Thus we find that

$$P(B < Y < R \mid E) = (1 + 3 + 6 + 10)/120 = 1/6$$

c) We see that

$$P(B < Y < R) = P(B < Y < R \mid E)P(E) + P(B < Y < R \mid E^c)P(E^c),$$

Since $P(B < Y < R \mid E^c) = 0$, from the above we have that $P(B < Y < R) = (1/6)/(5/9) = 5/54$.

24. a) Let E be the event that both balls are gold and F the event that at least one ball is gold. The probability we desire to compute is then $P(E \mid F)$. Using the definition of conditional probability we have that

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{P(\{G, G\})}{P(\{G, G\}, \{G, B\}, \{B, G\})} = \frac{1/4}{(1/4 + 1/4 + 1/4)} = 1/3$$

b) Since now the balls are mixed together in the urn, the difference between the pair G, B and B, G is no longer present. Thus we really have two cases to consider, either both balls are gold or one ball is gold and the other is black. Thus to have a second ball be gold will occur once out of these two choices and our probability is then 1/2.

35. Let U be the event that the present is upstairs, and let M be the event it was hidden by mom.

(a) $P(U) = P(U \mid M)P(M) + P(U \mid M^c)P(M^c) = .7(.6) + .5(.4) = .62$

(b) $P(M^c \mid U^c) = \frac{P(dad|down)P(dad)}{1-0.62} = \frac{(0.5)0.4}{0.38} = 10/19$

38. Let W and B represent the events of drawing a white ball or a black respectively, and let H and T denote the event of obtaining a head or a tail when we flip the coin. As stated in the problem when the outcome of the coin flip is heads (event H) a ball is selected from urn A. This urn has 5 white and 7 black balls. Thus $P(W \mid H) = 5/12$. Similarly, when the coin flip results in tails a ball is selected from urn B, which has 3 white and 12 black balls. Thus $P(W \mid T) = 3/12$. We would like to compute $P(T \mid W)$. Using Bayes formula we have

$$P(T \mid W) = \frac{P(W \mid T)P(T)}{P(W \mid T)P(T) + P(W \mid H)P(H)} = \frac{(3/15).(1/2)}{(3/15).(1/2) + (4/15).(1/2)} = 12/37.$$

1 Extra problems

1. (a) RR, RG, GR, GG
   (b) $P(RR) = P(\text{red 1st})P(\text{red 2nd} \mid \text{red 1st}) = 3/5 \times 2/4 = 6/20 = 0.3$
   (c) $P(RG) = 3/5 \times 2/4 = 6/20 = 0.3, P(GR) = 2/5 \times 3/4 = 6/20 = 0.3, P(GG) = 2/5 \times 1/4 = 2/20 = 0.1$
   (d) $P(A) = P(RR) = 0.3, P(B) = P(GG) = 0.1, P(C) = P(RG) + P(RR) = 0.3 + 0.3 = 0.6$
   (e) $P(A \mid B) = P(A \cap B)/P(B) = P(\text{Both balls are red AND Both balls are green})/P(B) = 0/0.1 = 0$, same for $P(B \mid A)$.
   (f) Mutually exclusive
   (g) $P(A \mid C) = P(A \cap C)/P(C) = P(\text{Both balls are red AND First ball is red})/P(C) = P(RR)/P(C) = 0.3/0.6 = 0.5, P(C \mid A) = P(C \cap A)/P(A) = P(\text{First ball is red AND Both balls are red})/P(A) = P(RR)/P(A) = 0.3/0.3 = 1.0$
   (h) A and C are dependent

2. (a) 00, 01, 10, 11, 05, 50, 55, 15,51
   (b) $P(11) = P(55) = P(15) = P(51) = 1/6 \times 1/6 = 1/36$, $P(00) = 4/6 \times 4/6 = 16/36 = 4/9$, $P(01) = P(10) = P(50) = P(05) = 4/6 \times 1/6 = 4/36 = 1/9$
   (c) $P(A) = P(10) + P(11) + P(15) = 4/36 + 1/36 + 1/36 = 6/36 = 1/6$, $P(B) = P(50) + P(51) + P(55) = 4/36 + 1/36 + 1/36 = 6/36 = 1/6$, $P(C) = P(10) + P(50) + P(00) = 4/36 + 4/36 + 16/36 = 24/36 = 4/6 = 2/3$
   (d) $P(A \cap B) = 0$ because these two events have no sample points in common and so $P(A \cup B) = P(A) + P(B) = 1/6 + 1/6 = 2/6 = 1/3$
   (e) They are mutually exclusive
   (f) $P(A \cap C) = P(\text{1st die is a one AND 2nd die is a zero}) = P(10) = 4/36 = 1/9$, $P(A \cup C) = P(A) + P(C) - P(A \cap C) = 6/36 + 24/36 - 4/36 = 26/36 = 13/18$
   (g) Independent because $P(A \mid C) = P(A \cap C)/P(C) = (1/9)/(2/3) = 3/18 = 1/6 = P(A)$, and $P(C \mid A) = P(A \cap C)/P(A) = (1/9)/(1/6) = 6/9 = 2/3 = P(C)$