STA 5327 HW #5 (Chpter 7)

#7.1 Suppose $X = (0, 1, 2, 3, 4)$ is discrete random variable with pmf $f(x|\theta)$, where $\theta \in \{1, 2, 3\}$.

Find the MLE of θ.

since $\hat{\theta}(x) = \sup_{\theta} L(\theta|x)$ = maximizing $f(x_i|\theta)$ for x_i w.r.t. θ
$(i = 1, 2, \cdots, 5)$

	x_1	x_2	x_3	x_4	x_5
x	0	1	2	3	4
$\hat{\theta}_{mle}$	1	1	2 & 3	3	3

#7.4 Prove the assertion in Example 7.2.8. That is, prove that $\hat{\theta}$ given there is the MLE when the range of θ is restricted to the positive axis.

Let $X_1, \dots, X_n$ be iid $N(\theta, 1)$. Then the joint pdf can be factored as

$\boxed{\text{refer to (Example 6.2.7) on page 277}}$

$$\prod_{i=1}^{n} f(x_i|\theta) = \underbrace{(2\pi)^{-n/2} \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{2}\right)}_{h(x)} \underbrace{\exp\left(-\frac{n(\bar{x}-\theta)^2}{2}\right)}_{g(T=\bar{x}|\theta)}$$

Setting $K = h(x)$, rewrite the likelihood function as

$$L(\theta|X) = K \exp\left(\frac{-n(\bar{x}-\theta)^2}{2}\right)$$

The log-likelihood is

$$l(\theta|\mathbf{x}) = \log K - \frac{n(\bar{x}-\theta)^2}{2}$$

The MLE is the value of $\theta \in \Theta$ which maximizes $l(\theta|\mathbf{x})$, where $\Theta = \{\theta : \theta \geq 0\}$.

In other words, $\frac{n(\bar{x}-\theta)^2}{2}$ is minimized.

$$l'(\theta) = \frac{l(\theta)}{\partial\theta} = n(\bar{x}-\theta)$$

For $\theta \geq 0$,

case I) if $\bar{x} \geq 0$, then $\theta := \bar{x}$ lies in Θ and $\bar{x}$ is the MLE.

case II) if $\bar{x} < 0$, then $\theta = \bar{x}$ lies outside Θ. In this case of $\bar{x} < 0$, the $l(\theta|x)$ is a decreasing function ($\because l'(\theta) < 0$) under the restricted range of θ to $\Theta = \{\theta : \theta \geq 0\}$. Thus, the maximum in Θ is at $\theta = 0$ which the MLE. Hence the MLE is

$$\begin{cases} \hat{\theta} = \bar{x} & \text{when } \bar{x} \geq 0 \\ \hat{\theta} = 0 & \text{when } \bar{x} < 0. \end{cases}$$

#7.6 Let $X_1, \ldots, X_n$ be a random sample from the pdf

$$f(x|\theta) = \theta x^{-2}, \quad 0 < \theta \leq x < \infty$$

(a) What is a sufficient statistic for θ?

$$\prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta x_i^{-2} I_{[\theta,\infty)}(x_i)$$

$$= \underbrace{\theta^n \prod_{i=1}^{n} x_i^{-2} I_{[\theta,\infty)}(x_{(1)})}_{g(T=x_{(1)}|\theta)} \cdot \underbrace{1}_{h(x)}$$ because $\theta \leq x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)} < \infty$ s.t. $\theta^{-2} \geq x_{(1)}^{-2} \geq \cdots > x_{(n)}^{-2} > 0$

By Factorization Theorem, $T = x_{(1)}$ is a S.S. for θ.

(b) Find the MLE of θ.

$$L(\theta|X) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n \prod_{i=1}^{n} x_i^{-2} I_{[\theta,\infty)}(x_{(1)})$$

Since $\hat{\theta}_{mle} = \arg\max_{\theta} L(\theta|X)$, so we can notice that if x_i is as small as possible, $L(\theta|X)$ is maximized, but $L(\theta|X) = 0$ if $0 < x_{(1)} < \theta$.

$$L(\theta|x) = \begin{cases} \theta^n \prod_{i=1}^{n} x_i^{(-2)} & \text{for } 0 < \theta \leq x_{(1)} \\ 0 & \text{for } x_{(1)} < \theta \end{cases}$$

Therefore $\hat{\theta}_{mle} = x_{(1)}$.

(c) Find the method of Moments estimator of θ.

$$m_1 = EX = \int_{\theta}^{\infty} x f(x|\theta)\,dx = \int_{\theta}^{\infty} \theta x^{-1}\,dx = \theta \log x \Big|_{\theta}^{\infty} = \infty$$

Thus, the method of Moments estimator θ does not exist.

#7.7 Let $X_1, \cdots, X_n$ be iid with one of two pdfs.

if $\theta = 0$, $f(x|\theta) = \begin{cases} 1, & 0<x<1 \\ 0, & \text{o.w.} \end{cases}$

if $\theta = 1$, $f(x|\theta) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0<x<1 \\ 0, & \text{o.w.} \end{cases}$

find the MLE.

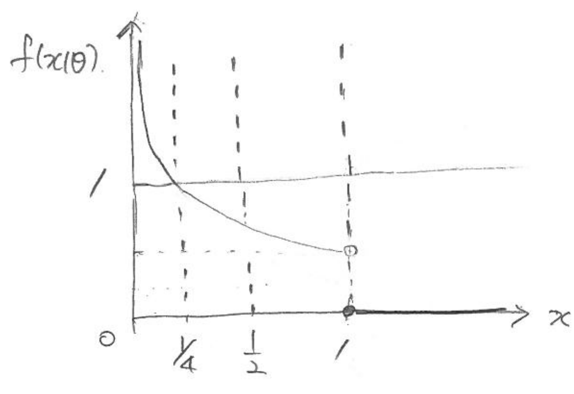
$$L(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} \{f(x_i|\theta)\}^{1-\theta} \{f(x|\theta)\}^{\theta} I_{(0,1)}(x_i)$$

$$= \prod_{i=1}^{n} 1^{1-\theta} \left(\frac{1}{2\sqrt{x_i}}\right)^{\theta} = \frac{1}{\left(\prod_{i=1}^{n} 2\sqrt{x_i}\right)^{\theta}} = \left(\prod_{i=1}^{n} 2\sqrt{x_i}\right)^{-\theta} = \left(2^n \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{2}}\right)^{-\theta}$$

$(0<x_i<1)$

For which values of $T(x) = \prod_{i=1}^{n} x_i$, $0<x_i<1$.

where



$$\hat{\theta}(x) = \sup_{\theta} L(\theta|x) = \max f(x|\theta)$$

$$\sup L(\theta|x) = \begin{cases} 1, & 2^n \left(\prod_{i=1}^{n} x_i\right)^{1/2} \geq 1 \Rightarrow \prod_{i=1}^{n} x_i \geq \frac{1}{2^{2n}} \\ 2^{-n}\left(\prod_{i=1}^{n} x_i\right)^{-1/2}, & 2^n \left(\prod_{i=1}^{n} x_i\right)^{1/2} < 1 \Rightarrow \prod_{i=1}^{n} x_i < \frac{1}{2^{2n}} \end{cases}$$

Thus, $\hat{\theta}_{mle} = 0 \cdot I_{[\frac{1}{2^{2n}}, \infty)}^{T(x)} + 1 \cdot I_{(-\infty, \frac{1}{2^{2n}})}^{T(x)}$

, where $T(x) = \prod_{i=1}^{n} x_i$, $\forall\ 0 < x_i < 1$

Therefore, $\hat{\theta}_{mle} = I_{(-\infty, \frac{1}{2^{2n}})}^{T(x)}$

#7.8. One observation, X, is taken from a $n(0, \sigma^2)$ population.

(a) Find an unbiased estimator of σ^2

since $X \sim n(0, \sigma^2)$, we know $EX = \mu = 0$

$$\sigma^2 = EX^2 - (EX)^2 = EX^2 - 0$$

$$\therefore \sigma^2 = EX^2$$

Thus, X^2 is an unbiased estimator of σ^2.

(b) Find the MLE of σ.

$$L(\sigma|x) = f(x|\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}x^2}$$

Taking log, then we have

$$l(\sigma) = \log L(\sigma|x) = -\frac{1}{2}\log(2\pi) - \log\sigma - \frac{1}{2\sigma^2}x^2$$

$$\frac{\partial l(\sigma)}{\partial\sigma} = -\frac{1}{\sigma} + \frac{x^2}{\sigma^3}$$

set $\frac{\partial l(\sigma)}{\partial\sigma} = -\frac{1}{\sigma} + \frac{x^2}{\sigma^3} = 0$.

$(\sigma > 0)$ $\quad \hat\sigma^2 = x^2 \Rightarrow \hat\sigma_{mle} = \sqrt{x^2} = |x|$ $\quad$ (local minimum)

For checking global maximum, ~~take second derivative~~. observe that $\frac{\partial l(\sigma)}{\partial\sigma} = \frac{1}{\sigma}\left[\frac{x^2}{\sigma^2} - 1\right]$ is > 0 for all $\sigma < |x|$ and < 0 $\forall$ $\sigma > |x|$. So by condition (iii) in class,

$$\hat\sigma_{MLE} = |x|$$

~~$\frac{\partial^2 l(\sigma)}{\partial\sigma^2} = \frac{1}{\sigma^2} - \frac{3x^2}{\sigma^4}$.~~

~~But when $\hat\sigma = |x|$, $\left.\frac{\partial^2 l(\sigma)}{\partial\sigma^2}\right|_{\hat\sigma=|x|} = \frac{1}{\sigma^2} - \frac{3}{\sigma^2} = -\frac{2}{\sigma^2} < 0$~~

~~Thus, $\hat\sigma_{mle} = |x|$ is a global minimum.~~

7.8

(c) Discuss how the method of moments estimator of σ might be found.

Note: μ_k = (population k-th moment) = EX^k

$\hat{\mu}_k = m_k$ = (sample k-th moment) = $\frac{1}{n}\sum_{i=1}^{n} x_i^k$

Given $X \sim n(0, \sigma^2)$.

$m_1 = EX = \mu = 0$ by given

$m_2 = EX^2 = \text{Var}X + m_1^2 = \sigma^2 = \sum_{i=1}^{n} \frac{X_i^2}{n}$.

$\Rightarrow \quad \sigma^2 = X^2$

$\hat{\sigma} = |X|$

7.9 Let $X_1, \cdots, X_n$ be iid with pdf

$f(x|\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta, \quad \theta > 0.$

Estimate θ using both the method of moments and maximum likelihood. Calculate the means and variances of the two estimators. Which one should be preferred and why?

First, we estimate θ using the method of moments.

From the pdf, we can notice that

$X \sim \text{Uniform}(0, \theta) \qquad (\theta > 0, \quad 0 \le x \le \theta)$.

Then $\hat{\mu} = EX = \frac{(0+\theta)}{2} = \frac{\theta}{2}$

$\hat{\mu} = \bar{X} = \frac{\theta}{2} \Rightarrow \tilde{\theta} = 2\bar{X}$. (denote $\tilde{\theta}$ as MME of θ)

$E(\tilde{\theta}) = E(2\bar{x}) = 2E(\bar{x}) = 2EX = 2\left(\frac{\theta}{2}\right) = \theta$

$\text{Var}(\tilde{\theta}) = \text{Var}(2\bar{x}) = 4\,\text{Var}(\bar{x}) = 4 \cdot \frac{1}{n} \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3n}$

Second, we estimate θ using maximum likelihood.

For $\theta > 0$,

$$L(\theta|x) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 \le x_i \le \theta) = \theta^{-n} I(0 \le x_{(1)}) I(x_{(n)} \le \theta)$$

$$= \begin{cases} \theta^{-n} & \text{for } \theta \ge x_{(n)} \\ 0 & \text{for } \theta < x_{(n)} \end{cases}$$

Thus $\hat{\theta}_{mle} = x_{(n)}$

The pdf of $\hat{\theta} = x_{(n)}$ (order statistic) $= \dfrac{n x^{n-1}}{\theta^n} \quad (0 \le x \le \theta)$.

Note: since $f_{X_{(j)}}(x) = \dfrac{n!}{(j-1)!(n-j)!} f_X(x) \left[F_X(x)\right]^{j-1} \left[1 - F_X(x)\right]^{n-j}$, so

for $j = n$, $f_{X_{(j)}}(x) = n \dfrac{1}{\theta} \left(\dfrac{x}{\theta}\right)^{n-1} \cancel{\left(1 - \dfrac{x}{\theta}\right)^{n-n}}$ 1

$$= \frac{n x^{n-1}}{\theta^n} \quad (0 \le x \le \theta)$$

$$E(\hat{\theta}) = E(x_{(n)}) = \int_0^\theta x \frac{n x^{n-1}}{\theta^n} dx = \frac{n}{\theta^n} \cdot \frac{x^{n+1}}{n+1}\Big|_0^\theta = \frac{n\theta}{n+1}$$

$$Var(\hat{\theta}) = Var(x_{(n)}) = E(x_{(n)}^2) - \left[E(x_{(n)})\right]^2 = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = \frac{n\theta^2(n^2+2n+1-n^2-2n)}{(n+2)(n+1)^2}$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2}$$

$$\left(\because E(x_{(n)}^2) = \int_0^\theta x^2 \frac{n x^{n-1}}{\theta^n} dx = \frac{n}{\theta^n} \frac{x^{n+2}}{n+2}\Big|_0^\theta = \frac{n\theta^2}{n+2} \right)$$

#7.9 (continue)

Since $E(\tilde{\theta}) = \tilde{\theta}$, $\tilde{\theta}$ (MOM estimator of θ) is an unbiased, while $\hat{\theta}$ (MLE of θ) is biased because $E(\hat{\theta}) = \frac{n\theta}{n+1}$. But as $n \to \infty$, $\frac{n}{n+1} \to 1$

So $E(\hat{\theta}) \to \theta$ as $n \to \infty$. In terms of precision measurement,

$$MSE(\hat{\theta}) = \left(\frac{n}{n+1}\theta - \theta\right)^2 + \frac{n\theta^2}{(n+2)(n+1)^2} = \frac{\theta^2}{(n+1)^2} + \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{\theta^2}{(n+1)(n+2)} \le \frac{\theta^2}{3n} = Var(\tilde{\theta})$$ and

$$Var(\tilde{\theta}) = \frac{\theta^2}{3n} > \frac{n\theta^2}{(n+2)(n+1)^2} = Var(\hat{\theta})$$ for all θ. Thus, the MLE is preferable to MOM estimate when n is large.

#7.10 The independent random variables $X_1, \dots, X_n$ have

$$P(X_i \le x \mid \alpha, \beta) = \begin{cases} 0 & \text{if } x < 0 \\ \left(\frac{x}{\beta}\right)^\alpha & \text{if } 0 \le x \le \beta \\ 1 & \text{if } x > \beta \end{cases}$$

(a) Find a two-dimensional sufficient statistic for (α, β).

$$F_X(x_i) = P(x_i \le x) = \left(\frac{x}{\beta}\right)^\alpha$$

$$f_X(x_i) = \alpha . \frac{x^{\alpha-1}}{\beta^\alpha}$$

$$\prod_{i=1}^n f_X(x_i) = \prod_{i=1}^n \left(\frac{\alpha}{\beta^\alpha}\right) x_i^{\alpha-1}\, I(0 \le x_i \le \beta)$$

$$= \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^n x_i\right)^{\alpha-1} I(0 \le x_{(1)})\, I(x_{(n)} \le \beta)$$

By factorization, $T = \left(\prod_{i=1}^n x_i,\ x_{(n)}\right)$ are sufficient.

#7.10

(b) Find the MLEs of α and β.

$$L(\alpha,\beta|\mathbf{x}) = \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^n x_i\right)^{\alpha-1} I(0 \le x_{(1)})\, I(x_{(n)} \le \beta).$$

log-likelihood is

$$\ell(\alpha,\beta) = n\log\alpha - n\alpha\log\beta + (\alpha-1)\sum_{i=1}^n \log x_i.$$

For any fixed α,

$L(\alpha,\beta|x) = 0$ if $\beta < x_{(n)}$ and

$L(\alpha,\beta|x)$ is a decreasing function if $\beta \ge x_{(n)}$ because $\frac{\partial \ell}{\partial\beta} = \frac{-n\alpha}{\beta} < 0$.

Thus the mle of β, $\hat\beta = x_{(n)}$.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - n\log\beta + \sum_{i=1}^n \log x_i = 0 \quad \text{and} \quad \frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n}{\alpha^2} < 0 \quad (\text{maximum})$$

$$\hat\alpha = \frac{n}{n\log\hat\beta - \sum_{i=1}^n \log x_i}$$

By substituting $\hat\beta = x_{(n)}$, we have

$$\hat\alpha = \frac{n}{n\log x_{(n)} - \sum_{i=1}^n \log x_i} = \left[\frac{1}{n}\sum_{i=1}^n\left(\log x_{(n)} - \log x_i\right)\right]^{-1}$$

Thus the mle of α, $\hat\alpha = \left[\frac{1}{n}\sum_{i=1}^n\left(\log x_{(n)} - \log x_i\right)\right]^{-1}$

(c) For given data, we obtain

$$\left.\begin{aligned} x_{(n)} &= 25.0 \\ \log\prod_{i=1}^n x_i &= \sum_{i=1}^n \log x_i = 43.95 \end{aligned}\right\} \Rightarrow \hat\alpha = 12.59 \text{ and } \hat\beta = 25.0$$

#7.11 Let $X_1, \dots, X_n$ be iid with pdf

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 \le x \le 1, \quad 0 < \theta < \infty$$

$$L(\theta|x) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} I(0 \le x_i \le 1)$$

$$l(\theta) = \log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$$

(a) Find the MLE of θ, and show that its variance $\to 0$ as $n \to \infty$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i = 0$$

$$\hat{\theta} = \left(-\frac{\sum_{i=1}^{n}\log x_i}{n}\right)^{-1}$$

$$\frac{\partial^2 l}{\partial \theta^2} = -\frac{n}{\theta^2} < 0 \quad \text{(local max)}$$

Thus $\hat{\theta} = \left(-\frac{\sum_{i=1}^{n}\log x_i}{n}\right)^{-1}$ is mle.

Set $Y_i = -\log X_i$.

$$g^{-1}(y_i) = e^{-y_i} \qquad \frac{d}{dy} g^{-1}(y_i) = -e^{-y_i}$$

$$f_{Y_i}(y_i) = f_X(g^{-1}(y_i)) \left| \frac{d}{dy} g^{-1}(y_i) \right|$$

$$= \theta (e^{-y_i})^{\theta-1} \left| e^{-y_i} \right| = \theta e^{-\theta y_i} \sim \exp\left(\tfrac{1}{\theta}\right)$$

Since $y_i \sim \exp(\frac{1}{\theta})$, then

$T = \sum y_i = -\sum_{i=1}^{n}\log x_i \sim \text{gamma}(n, \frac{1}{\theta})$.

But $\frac{1}{T} \sim$ inverted gamma and its pdf is $f(u) = \frac{\theta^n}{\Gamma(n)} u^{n-1} e^{-\theta u}$

Then $E(\frac{1}{T}) = \int_0^\infty \frac{1}{t} f(t)\,dt = \frac{\theta^n}{\Gamma(n)} \int_0^\infty t^{n-2} e^{-\theta t}\,dt$

$$= \frac{\theta^n}{\Gamma(n)} \cdot \frac{\Gamma(n-1)}{\theta^{n-1}} \underbrace{\int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} t^{n-2} e^{-\theta t}\,dt}_{1}$$

$$= \frac{\theta}{(n-1)}.$$

#7.11 (continue)

$$E\left(\frac{1}{T^2}\right) = \int_0^\infty \frac{1}{t^2} f(t)\,dt = \frac{\theta^n}{\Gamma(n)} \int_0^\infty t^{n-3} e^{-\theta t}\,dt$$

$$= \frac{\theta^n}{\Gamma(n)} \cdot \frac{\Gamma(n-2)}{\theta^{n-2}} \underbrace{\int_0^\infty \frac{\theta^{n-2}}{\Gamma(n-2)} t^{n-3} e^{-\theta t}\,dt}_{1}$$

$$= \frac{\theta^2}{(n-1)(n-2)}.$$

$$E(\hat{\theta}) = E\left(\frac{n}{T}\right) = n\,E\left(\frac{1}{T}\right) = \frac{n\theta}{n-1}$$

$$Var(\hat{\theta}) = Var\left(\frac{n}{T}\right) = n^2\,Var\left(\frac{1}{T}\right) = n^2\left[E\left(\frac{1}{T^2}\right) - \left(E\left(\frac{1}{T}\right)\right)^2\right]$$

$$= n^2\left[\frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2}\right]$$

$$= \frac{n^2\theta^2}{(n-1)}\left[\frac{1}{(n-2)} - \frac{1}{(n-1)}\right]$$

$$= \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

Thus $Var(\hat{\theta}) = \frac{n^2\theta^2}{(n-1)^2(n-2)} \to 0$ as $n \to \infty$

(b) Find the method of moments estimator of θ.

We notice that $X \sim Beta(\theta, 1)$.

So $E(X) = \frac{\Sigma x_i}{n} = \frac{\theta}{\theta+1}$

$$\therefore \tilde{\theta}_{MOM} = \frac{\Sigma X_i}{n - \Sigma X_i}$$

#7.12 Let $X_1, \dots, X_n$ be a random sample from a population with pmf

$$P_\theta(X=x) = \theta^x (1-\theta)^{1-x}, \quad x = 0 \text{ or } 1, \quad 0 \le \theta \le \tfrac{1}{2}.$$

(a) Find the method of moments estimator and MLE of

Since $X \overset{iid}{\sim} \text{Bernoulli}(\theta)$,

$$EX = \frac{1}{n}\Sigma X_i = \theta$$

$$\therefore \tilde{\theta}_{MOM} = \frac{\Sigma X_i}{n}$$

$$L(\theta|x) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i} \quad (x_i = 0 \text{ or } 1).$$

Let $y = \Sigma x_i$.

$$l(\theta) = \log L(\theta) = y\log\theta + (n-y)\log(1-\theta)$$

$$\frac{\partial l(\theta)}{\partial\theta} = \frac{y}{\theta} - \frac{n-y}{1-\theta} = 0 \quad \text{and} \quad -\frac{y}{\theta^2} - \frac{(n-y)}{(1-\theta)^2} < 0. \text{ (local max)}$$

$$y - \cancel{y\theta} = n\theta - \cancel{y\theta}$$

$$\hat{\theta}_{mle} = \frac{y}{n} = \frac{\Sigma X_i}{n} = \bar{x}$$

But $0 \le \theta \le \frac{1}{2}$ by given, so if $\bar{x} \le \frac{1}{2}$, then $\hat{\theta}_{mle} = \bar{x}$; otherwise $\hat{\theta}_{mle} = \frac{1}{2}$.
Thus the MLE of θ, $\hat{\theta}_{mle} = \min\{\bar{x}, \frac{1}{2}\}$.

(b) Find the mean squared errors of each of the estimators

$$MSE(\tilde{\theta}_{MOM}) = Var\,\tilde{\theta}_{MOM} + \underbrace{[E(\tilde{\theta}_{MOM} - \theta)]^2}_{bias^2} = \frac{\theta(1-\theta)}{n} + 0 = \frac{\theta(1-\theta)}{n}$$

$$MSE(\hat{\theta}_{mle}) = E(\hat{\theta}_{mle} - \theta)^2 = \sum_{y=0}^{n}(\hat{\theta}-\theta)^2\binom{n}{y}\theta^y(1-\theta)^{n-y}. \quad (y = \Sigma x_i \sim \text{Binomial}(n,\theta))$$

$$= \sum_{y=0}^{[\frac{n}{2}]}\left(\frac{y}{n}-\theta\right)^2\binom{n}{y}\theta^y(1-\theta)^{n-y} I\left(\tfrac{y}{n} \le \tfrac{1}{2} \text{ and } n \text{ is even}\right) + \sum_{y=[\frac{n}{2}]+1}^{n}\left(\frac{1}{2}-\theta\right)^2\binom{n}{y}\theta^y(1-\theta)^{n-y} I_{\left(\frac{y}{n} > \frac{1}{2} \text{ and } n \text{ is odd}\right)}$$

#7.12 (continue)

(c) Which estimator is prefered? Justify your choice.

For $0 < \theta \le \frac{1}{2}$,

$$
\begin{aligned}
& MSE(\tilde{\theta}) - MSE(\hat{\theta}) \\
&= \sum_{y=0}^{n}\left(\frac{y}{n}-\theta\right)^2\binom{n}{y}\theta^y(1-\theta)^{n-y} - \Bigg[\underbrace{\sum_{y=0}^{\lceil \frac{n}{2}\rceil}\left(\frac{y}{n}-\theta\right)^2\binom{n}{y}\theta^y(1-\theta)^{n-y}}_{\text{when } n \text{ is even } \& \ \frac{y}{n}\le\frac{1}{2}} + \underbrace{\sum_{y=\lceil\frac{n}{2}\rceil+1}^{n}\left(\frac{1}{2}-\theta\right)^2\binom{n}{y}\theta^y(1-\theta)^{n-y}}_{\text{when } n \text{ is odd } \& \ \frac{y}{n}>\frac{1}{2}}\Bigg] \\
&= \sum_{y=\lceil\frac{n}{2}\rceil+1}^{n}\left[\left(\frac{y}{n}-\theta\right)^2-\left(\frac{1}{2}-\theta\right)^2\right]\binom{n}{y}\theta^y(1-\theta)^{n-y} \\
&= \sum_{y=\lceil\frac{n}{2}\rceil+1}^{n}\left[\left(\frac{y}{n}-\theta+\frac{1}{2}-\theta\right)\left(\frac{y}{n}\cancel{-\theta}-\frac{1}{2}\cancel{+\theta}\right)\right]\binom{n}{y}\theta^y(1-\theta)^{n-y} \\
&= \sum_{y=\lceil\frac{n}{2}\rceil+1}^{n}\left[\left(\frac{y}{n}+\frac{1}{2}-2\theta\right)\left(\frac{y}{n}-\frac{1}{2}\right)\right]\binom{n}{y}\theta^y(1-\theta)^{n-y}\cdot I\left(\frac{y}{n}>\frac{1}{2}\right)
\end{aligned}
$$

We know $\left(\frac{y}{n}+\frac{1}{2}-2\theta\right)\left(\frac{y}{n}-\frac{1}{2}\right) > 0$ since $0<\theta\le\frac{1}{2}$ and $\frac{y}{n}>\frac{1}{2}$.

Thus, $MSE(\tilde{\theta}_{MOM}) > MSE(\hat{\theta}_{mle})$.

Hence $\hat{\theta}_{mle}$ is prefered.

#7.13. Let $X_1, \cdots, X_n$ be a sample from a population with double exponential pdf

$$f(x|\theta) = \frac{1}{2}e^{-|x-\theta|}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

Find the MLE of θ. (Hint : Consider the case of even n separate from that of odd n, and find the MLE in terms of the order statistics. A complete treatment of this problem is given in Norton 1984.)

$$L(\theta|x) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i-\theta|}$$

$$= \frac{1}{2^n} e^{-\sum_{i=1}^{n}|x_i-\theta|}$$

$$l(\theta) = \log L(\theta|x) = -n\log 2 - \sum_{i=1}^{n}|x_i-\theta|$$

But $\sum_{i=1}^{n}|x_i-\theta| = \sum_{i=1}^{n}|x_{(i)}-\theta|$, where $x_{(i)}$ is order statistics ($x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$)

For $x_{(j)} \le \theta \le x_{(j+1)}$,

$$\sum_{i=1}^{n}|x_{(i)}-\theta| = \sum_{i=1}^{j}(\theta - x_{(i)}) + \sum_{i=j+1}^{n}(x_{(i)}-\theta)$$

$$= \sum_{i=1}^{j}\theta - \sum_{i=1}^{j}x_{(i)} + \sum_{i=j+1}^{n}x_{(i)} - \sum_{i=j+1}^{n}\theta$$

But if $j = \frac{n}{2}$, $2j - n = 0$ if n is even.

$$= (2j-n)\theta - \sum_{i=1}^{j}x_{(i)} + \sum_{i=j+1}^{n}x_{(i)}$$

And this function of θ decreases for $j < \frac{n}{2}$ and increases for $j > \frac{n}{2}$.

Thus the mle of θ is the midpoint of $[x_{(\frac{n}{2})}, x_{(\frac{n}{2})+1}]$.

Hence $\hat{\theta}_{mle} = x_{(\frac{n+1}{2})}$.