HW5 solutions

4.14 $P\{X=0\} = P\{1 \text{ loses to } 2\} = \boxed{\frac{1}{2}}$

$P\{X=1\} = P\{1 \text{ loses to } 3,\ 1 \text{ wins to } 2\} = \frac{1}{3}\cdot\frac{1}{2} = \boxed{\frac{1}{6}}$

$P\{X=2\} = P\{1 \text{ loses to } 4,\ 1 \text{ wins to } 3\ (1 \text{ wins to } 2)\} = \frac{1}{4}\cdot\frac{1}{3} = \boxed{\frac{1}{12}}$

This condition must happen if '1 wins to 3' happens, so we do not need to times $\frac{1}{2}$

$P\{X=3\} = P\{1 \text{ loses to } 5,\ 1 \text{ wins to } 4\} = \frac{1}{5}\cdot\frac{1}{4} = \boxed{\frac{1}{20}}$

$P\{X=4\} = P\{1 \text{ wins all players}\} = \boxed{\frac{1}{5}}$

4.17. (a) $P\{X=1\} = P\{X\le 1\} - P\{X<1\} = \left(\frac{1}{2}+\frac{b-1}{4}\right) - \left(\frac{b}{4}\right)\Big|_{b=1} = \boxed{\frac{1}{4}}$

$P\{X=2\} = P\{X\le 2\} - P\{X<2\} = \frac{11}{12} - \left(\frac{1}{2}+\frac{b-1}{4}\right)\Big|_{b=2} = \boxed{\frac{1}{6}}$

$P\{X=3\} = P\{X\le 3\} - P\{X<3\} = 1 - \frac{11}{12} = \boxed{\frac{1}{12}}$.

(b) $P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \lim_{n\to\infty} P\left\{X \le \frac{3}{2} - \frac{1}{n}\right\} - P\left\{X \le \frac{1}{2}\right\}$

$= \frac{1}{2} + \frac{\frac{3}{2}-1}{4} - \frac{\frac{1}{2}}{4}$

$= \boxed{\frac{1}{2}}$

4.19. $F(b)=\sum_{x\le b} P(x)$, so

$P(0)=P\{X\le 0\}-P\{X<0\}=\frac{1}{2}-0=\boxed{\frac{1}{2}}$

$P(1)=P\{X\le 1\}-P\{X<1\}=\frac{3}{5}-\frac{1}{2}=\boxed{\frac{1}{10}}$

$P(2)=P\{X\le 2\}-P\{X<2\}=\frac{4}{5}-\frac{3}{5}=\boxed{\frac{1}{5}}$

$P(3)=P\{X\le 3\}-P\{X<3\}=\frac{9}{10}-\frac{4}{5}=\boxed{\frac{1}{10}}$

$P(3.5)=P\{X\le 3.5\}-P\{X<3.5\}=1-\frac{9}{10}=\boxed{\frac{1}{10}}$

4.20. a) $P\left\{\substack{\text{(1st)}\\ \text{win}}\right\}=\frac{18}{38}$

$P\left\{\substack{\text{1st}\\ \text{lose}}\ \substack{\text{2nd}\\ \text{win}}\ \substack{\text{3rd}\\ \text{win}}\right\}=\frac{20}{38}\cdot\frac{18}{38}\cdot\frac{18}{38}$

$P\{X>0\}=P\left\{\substack{\text{1st}\\ \text{win}}\right\}+P\left\{\substack{\text{1st}\\ \text{lose}}\ \substack{\text{2nd}\\ \text{win}}\ \substack{\text{3rd}\\ \text{win}}\right\}$

$=\frac{18}{38}+\frac{20\cdot 18^2}{38^3}$

≈ 0.5918

b) In order to answer this question, we should do c) part first.

c) $P\left\{\substack{\text{1st}\\ \text{lose}}\ \substack{\text{2nd}\\ \text{lose}}\ \substack{\text{3rd}\\ \text{lose}}\right\}=\left(\frac{20}{38}\right)^3$

$P\left\{\substack{\text{1st}\\ \text{lose}}\ \substack{\text{2nd}\\ \text{win}}\ \substack{\text{3rd}\\ \text{lose}}\right\}=\frac{20}{38}\cdot\frac{18}{38}\cdot\frac{20}{38}=\frac{18\cdot 20^2}{38^3}$

$P\left\{\substack{\text{1st}\\ \text{lose}}\ \substack{\text{2nd}\\ \text{lose}}\ \substack{\text{3rd}\\ \text{win}}\right\}=\frac{20}{38}\cdot\frac{20}{38}\cdot\frac{18}{38}=\frac{18\cdot 20^2}{38^3}$

$EX=1\cdot P\{X>0\}+(-1)\cdot\left(P\left\{\substack{\text{1st}\\ \text{lose}}\ \substack{\text{2nd}\\ \text{win}}\ \substack{\text{3rd}\\ \text{lose}}\right\}+P\left\{\substack{\text{1st}\\ \text{lose}}\ \substack{\text{2nd}\\ \text{lose}}\ \substack{\text{3rd}\\ \text{win}}\right\}\right)+(-3)\cdot P\left\{\substack{\text{1st}\\ \text{lose}}\ \substack{\text{2nd}\\ \text{lose}}\ \substack{\text{3}\\ \text{lose}}\right.$

$=0.5918-0.2624-0.4374$

$=\boxed{-0.108}$

Now, come back to answer b).

This is $\boxed{\text{not}}$ a so called "winning" strategy at all, because its expectation is less than 0 dollar.

4.24. a) $E_B[A \text{ writes } 1] = 1\cdot p + (-\frac{3}{4})(1-p) = \boxed{\frac{7}{4}p - \frac{3}{4}}$

b) $E_B[A \text{ writes } 2] = 2\cdot(1-p) + (-\frac{3}{4})\cdot p = \boxed{2 - \frac{11}{4}p}$

When $E_B[A \text{ writes } 1] = E_B[A \text{ writes } 2]$, the value of p maximizes the minimum posible value of B's expected gain.

$\frac{7}{4}p - \frac{3}{4} = 2 - \frac{11}{4}p \Rightarrow p = \boxed{11/18}$ and $E = \frac{7}{4}\times\frac{11}{18} - \frac{3}{4} = \boxed{\frac{23}{72}} = \boxed{0.3194}$

c) $E_A[B \text{ chooses } 1] = (-1)\cdot q + \frac{3}{4}(1-q) = \boxed{\frac{3}{4} - \frac{7}{4}q}$

$E_A[B \text{ chooses } 2] = (-2)(1-q) + \frac{3}{4}\cdot q = \boxed{-2 + \frac{11}{4}q}$

As above, when $\frac{3}{4} - \frac{7}{4}q = -2 + \frac{11}{4}q \Rightarrow q = \boxed{\frac{11}{18}}$, $E = \frac{3}{4} - \frac{7}{4}\times\frac{11}{18} = \boxed{-\frac{23}{72}} = \boxed{-0.31}$

4.25. $P\{H_1\} = 0.6$, $P\{T_1\} = 0.4$, $P\{H_2\} = 0.7$, $P\{T_2\} = 0.3$

a) $P\{X=1\} = P\{H_1\}\cdot P\{T_2\} + P\{T_1\}\cdot P\{H_2\} = 0.6\cdot 0.3 + 0.4\cdot 0.7 = \boxed{0.46}$

b) $P\{X=0\} = P\{T_1\}\cdot P\{T_2\} = 0.4\cdot 0.3 = 0.12$

$P\{X=2\} = P\{H_1\}\cdot P\{H_2\} = 0.6\cdot 0.7 = 0.42$

$EX = 0\cdot P\{X=0\} + 1\cdot P\{X=1\} + 2\cdot P\{X=2\}$

$= 0 + 1\cdot 0.46 + 2\cdot 0.42 = \boxed{1.3}$

4.31. $E[\text{score}] = p^*[1-(1-p)^2] + (1-p^*)(1-p^2)$.

In order to get critical point, take the derivative of E about p

$$\frac{dE}{dp} = -2p^*(1-p)\cdot(-1) + (1-p^*)\cdot(-2p)$$

$$= 2p^* - 2p = 0$$

$$\Rightarrow \boxed{p = p^*}$$

4.38 a) $Var(X) = EX^2 - (EX)^2$

$$5 = EX^2 - 1^2 \Rightarrow EX^2 = 6$$

$$E[(2+X)^2] = E[4 + 4X + X^2]$$

$$= E[4] + 4EX + EX^2$$

$$= 4 + 4\times 1 + 6$$

$$= \boxed{14}$$

or, you can calculate like this.

$Var(2+X) = Var X = 5$

$E[2+X] = 2 + EX = 3$

$E[(2+X)^2] - (E[2+X])^2 = Var(2+X)$

$E[(2+X)^2] - 3^2 = 5 \Rightarrow E[(2+X)^2] = \boxed{14}$.

b) $Var(4+3X) = 3^2 Var(X)$

$$= 9\times 5$$

$$= \boxed{45}$$

4.41 Let X be the total number that the man is correct.

$P\{$at least as well as 7 correct$\}$

$= P\{X=7\} + P\{X=8\} + P\{X=9\} + P\{X=10\}$

$= \binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 + \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + \binom{10}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1 + \binom{10}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0$

$= (120+45+10+1)\times\left(\frac{1}{2}\right)^{10}$

$= \boxed{0.1719}$