①

STA5327 HW#6.

7.19 Suppose that the random variables $Y_1, \cdots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i \quad , \quad i=1, \cdots, n$$

where $x_1, \cdots, x_n$ are fixed constants, and $\epsilon_1, \cdots, \epsilon_n$ are iid $n(0, \sigma^2)$, σ^2 unknown.

(a) Find a two-dimensional sufficient statistic for (β, σ^2).

$$f(y_i|\beta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2\right)$$

Let $\theta = (\beta, \sigma^2)$

$$L(\theta|\mathbf{y}) = \prod_{i=1}^{n} f(y_i|\beta,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2\right)$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2\right)$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i^2 - 2\beta x_i y_i + \beta^2 x_i^2)\right)$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i y_i - \frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right)$$

$$= \underbrace{(2\pi\sigma^2)^{-\frac{n}{2}}}_{c(\theta)} \underbrace{\exp\left(-\frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right)}_{h(x)} \exp\underbrace{\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i y_i\right)}_{w_i(\theta) t_i(y)}$$

where $\theta = (\beta, \sigma^2)$. Then the statistic

$$T(\mathbf{Y}) = \left(\sum_{i=1}^{n} Y_i^2 \ , \ \sum_{i=1}^{n} x_i Y_i\right).$$

By Definition of sufficient statistic for θ (Defn 6.2.1),

$T(\mathbf{Y}) = \left(\sum_{i=1}^{n} Y_i^2, \ \sum_{i=1}^{n} x_i Y_i\right)$ is a sufficient statistic for $\theta = (\beta, \sigma^2)$.

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(b) Find the MLE of β and show that it is an unbiased estimator of β.

$$\ell(\theta|y) = \log L(\theta|y) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} Y_i^2 + \frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i Y_i - \frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n} x_i^2$$

For a fixed σ^2,

$$\frac{\partial \ell}{\partial \beta} = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i Y_i - \frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i^2 = 0$$

Then we have $\hat{\beta} = \dfrac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}$. (Notice that $\hat{\beta}$ doesn't depend on σ^2)

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{\sum x_i^2}{\sigma^2} < 0 \quad \text{(local maximum)}$$

Thus, $\hat{\beta}$ is the mle.

$$E\hat{\beta} = E\left(\frac{\sum x_i Y_i}{\sum x_i^2}\right) = \frac{\sum_{i=1}^{n} x_i\, EY_i}{\sum x_i^2} = \frac{\sum_{i=1}^{n} x_i(\beta x_i)}{\sum x_i^2} = \beta$$

Thus $\hat{\beta}$ is an unbiased estimator of β.

(c) Find the distribution of the MLE of β.

We know $\hat{\beta} = \dfrac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}$. We express $\hat{\beta} = \sum_{i=1}^{n} a_i Y_i$, where $a_i = \dfrac{x_i}{\sum_{i=1}^{n} x_i^2}$ are constants.

$$Var(\hat{\beta}) = Var\left(\sum_{i=1}^{n} a_i Y_i\right) = \sum_{i=1}^{n} a_i^2\, Var(Y_i) = \sum_{i=1}^{n} a_i^2 \sigma^2 = \sum_{i=1}^{n}\left(\frac{x_i}{\sum x_i^2}\right)^2 \sigma^2 = \frac{\sum x_i^2}{(\sum x_i^2)^2}\sigma^2 = \frac{\sigma^2}{\sum x_i^2}$$

Therefore, $E(\hat{\beta}) = \beta$ (from the result of the part (b)) and $Var(\hat{\beta}) = \dfrac{\sigma^2}{\sum x_i^2}$.

$$\hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right).$$

#7.20 Consider $Y_1, \dots, Y_n$ as defined in Exercise 7.19.

(a) Show that $\frac{\Sigma Y_i}{\Sigma x_i}$ is an unbiased estimator of β

$$E\left(\frac{\Sigma Y_i}{\Sigma x_i}\right) = \frac{\sum_{i=1}^{n} E(Y_i)}{\Sigma x_i} = \frac{\sum_{i=1}^{n} \beta x_i}{\Sigma x_i} = \frac{\beta \cancel{\Sigma x_i}}{\cancel{\Sigma x_i}} = \beta .$$

(b) Calculate the exact variance of $\Sigma Y_i / \Sigma x_i$ and compare it to the variances of the estimators in (Ex 7.

$$Var\left(\frac{\Sigma Y_i}{\Sigma x_i}\right) = \frac{\sum_{i=1}^{n} Var(Y_i)}{\left(\sum_{i=1}^{n} x_i\right)^2} = \frac{\sum_{i=1}^{n} \sigma^2}{\left(\sum_{i=1}^{n} x_i\right)^2} = \frac{n\sigma^2}{(\Sigma x_i)^2} = \frac{n\sigma^2}{(n\bar{x})^2} = \frac{\sigma^2}{n\bar{x}^2}$$

$(\because \Sigma x = n\bar{x})$

From the part (c), we obtained

$$Var(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

But $\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \geq 0 \Rightarrow \Sigma x_i^2 \geq n\bar{x}^2$.

Thus

$$\frac{\sigma^2}{n\bar{x}^2} \geq \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}.$$

which means that $Var(\hat{\beta})$ is smaller than $Var\left(\frac{\Sigma Y_i}{\Sigma x_i}\right)$.