HW 6 solutions

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4.49. $P_{1H}=0.4$, $P_{1T}=0.6$, $P_{2H}=0.7$, $P_{2T}=0.3$

(a) Each coin has $\frac{1}{2}$ chance to be selected.

P(heads 7 of 10 flips) = P(coin 1 heads 7 of 10 flips) + P(coin 2 heads 7 of 10 flips)

$$= \frac{1}{2}\times\binom{10}{7}(0.4)^7(0.6)^3 + \frac{1}{2}\binom{10}{7}(0.7)^7(0.3)^3$$

$$= \frac{1}{2}\times 120\times(0.4^7\,0.6^3 + 0.7^7\,0.3^3)$$

$$= 60\times 0.0026$$

$$= \boxed{0.156} \text{ or } \boxed{0.1546476}$$

(b) P(heads 7 of 10 flips | 1st headed)

$$= \binom{9}{6}\cdot(0.4)^6\cdot 0.6^3\cdot\frac{0.4\cdot\frac{1}{2}}{0.4\cdot\frac{1}{2}+0.7\cdot\frac{1}{2}} + \binom{9}{6}(0.7)^6\cdot 0.3^3\cdot\frac{0.7\cdot\frac{1}{2}}{0.4\cdot\frac{1}{2}+0.7\cdot\frac{1}{2}}$$

$$\cancel{=\boxed{0.108}}$$

$$= \boxed{0.1968}.$$

Detail

= P(7 Heads of 10 flips | 1st head, C_1) · P(C_1 | 1H) + P(7 heads of 10 flips | 1st H, C_2) · P(C_2 | 1H)

= P(6 H in 9 flips | C_1) · P(C_1 | 1H) + P(6 H in 9 flips | C_2) · P(C_2 | 1H)

$$P(C_1|1H) = \frac{P(1H|C_1)\cdot P(C_1)}{P(1H|C_1)\,P(C_1) + P(1H|C_2)\cdot P(C_2)} = \frac{0.4\cdot\frac{1}{2}}{0.4\cdot\frac{1}{2}+0.7\cdot\frac{1}{2}}.$$

P(C_2 | 1H) will use the same method to compute.

4.58

(a) Poisson.

$\lambda = np = 0.8$

$$P(X=2) = \frac{e^{-0.8}(0.8)^2}{2!}$$

$$= \boxed{0.14388}$$

Binomial

$$P(X=2) = \binom{8}{2} 0.1^2 \, 0.9^6$$

$$= \boxed{0.1488}$$

(b) Poisson

$\lambda = np = 9.5$

$$P\{X=9\} = \frac{e^{-\lambda}\lambda^i}{i!}$$

$$= \frac{e^{-9.5}\, 9.5^9}{9!}$$

$$= \boxed{0.13}$$

Binomial

$$P\{X=9\} = \binom{10}{9} \cdot 0.95^9 \cdot 0.05^1$$

$$= \boxed{0.3151}$$

(c) Poisson

$\lambda = 10 \times .1 = 1$

$$P\{X=0\} = \frac{e^{-\lambda} \cdot \lambda^i}{i!}$$

$$= \frac{e^{-1} 1^0}{0!}$$

$$= e^{-1}$$

$$= \boxed{0.3679}$$

Binomial

$$P\{X=0\} = \binom{10}{0} \cdot 0.1^0 \cdot (0.9)^{10}$$

$$= \boxed{0.3487}$$

(d) Poisson

$\lambda = 9 \times 0.2 = 1.8$

$$P\{X=4\} = \frac{e^{-\lambda}\lambda^i}{i!}$$

$$= \frac{e^{-1.8} \cdot 1.8^4}{4!}$$

$$= \boxed{\cancel{0.0446}} \quad \boxed{0.072}$$

Binomial

$$P\{X=4\} = \binom{9}{4} 0.2^4 \, 0.8^5$$

$$= \boxed{0.0661}$$

4.59 $X \sim \text{poisson}\left(\frac{50}{100}\right) = \text{poisson}\left(\frac{1}{2}\right)$.

(1) $P\{\text{at least once}\} = 1 - P\{\text{never win}\}$

$$= 1 - \frac{e^{-\lambda}\lambda^{i}}{i!}$$

$$= 1 - \frac{e^{-\frac{1}{2}}\cdot\left(\frac{1}{2}\right)^{0}}{0!}$$

$$= \boxed{1 - e^{-\frac{1}{2}}}$$

$$= \boxed{0.3935}$$

(2) $P\{\text{exactly once}\} = \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^{1}}{1!} = \boxed{\frac{1}{2}e^{-\frac{1}{2}}} = \boxed{0.3033}$

(3) $P\{\text{at least twice}\} = \cancel{P}\ 1 - P\{\text{never win}\} - P\{\text{exactly once}\}$

$$= P\{\text{at least once}\} - P\{\text{exactly once}\}$$

$$= 0.3935 - 0.3033$$

$$= \boxed{0.0902} \text{ or } \boxed{1 - e^{-\frac{1}{2}} - \frac{1}{2}e^{-\frac{1}{2}}}$$

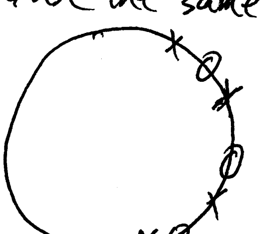
4.66. (a) We just need to arrange $(2n-1)$ people's seat, and the rest 1 seat will be arranged to the remaining person. So the population is $(2n-1)!$

We subtract 2 people (a couple) from $2n$, then arrang $(2n-2)$ people, we get $(2n-2)!$, then arrange back the couple, there are two possible seats, ♀♂ and ♂♀, so we get $2\cdot(2n-2)!$

$$P(C_i) = \frac{2(2n-2)!}{(2n-1)!} = \boxed{\frac{2}{2n-1}}$$

(b) $P(C_j|C_i) = \frac{P(C_j, C_i)}{P(C_i)}$, to arrange the two couples, we consider to arrange $\underset{\uparrow\ \text{one of } i}{2n-1} \underset{\uparrow\ \text{one of } j}{-1} = 2n-2$

Then, use the same trick as above, remove one person out there are $(2n-2-1) = (2n-3)$ persons.



The person has 4 possible seats, $\Rightarrow 4(2n-2)! \Rightarrow \cancel{P(C_j|C_i)}\ P(C_j, C_i) = \frac{4(2n-3)!}{(2n-1)!} = \frac{4}{(2n-1)(2n-2)}$

$$P(C_j|C_i) = \frac{P(C_j, C_i)}{P(C_i)} = \frac{\frac{4}{(2n-1)(2n-2)}}{\frac{2}{2n-1}} = \boxed{\frac{1}{n-1}}$$

(C). $P\{$No couples seat next to each other$\}$

$= 1 - P\{C_1 \cup C_2 \cup \cdots \cup C_n\}$

$= 1 - \left(\sum_{i=1}^{n} P(C_i) - \sum_{i<j} P(C_i, C_j) + \cdots\right)$

$= 1 - \left(\frac{2n}{2n-1} - \sum_{i<j} P(C_j | C_i) \cdot P(C_i) + \cdots\right)$

$= 1 - \left(\frac{2n}{2n-1} - \binom{n}{2} \frac{2}{(2n-1)(n-1)} + \cdots\right)$

When n is large, $\frac{1}{n-1} \approx \frac{2}{2n-1}$, which $\frac{2}{2n-1} = \frac{1}{n-\frac{1}{2}}$.

While C_i and C_j are dependent, the dependence is weak for large n.

By ~~possion~~ poisson paradim we can expect the number of couples sitting together to have a poisson approximation with rate $\lambda = n\left(\frac{2}{2n-1}\right) \approx 1$.

so $P\{N=0\} = e^{-1}$.

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4.4 $\sum_{i=1}^{\infty} P\{N \geq i\} = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P\{N=k\}$

$= \cancel{\sum_{k=i}^{\infty}} P\{N=1\} + P\{N=2\} + P\{N=3\} + \cdots + P\{N=2\} + P\{N=3\} + \cdots + P\{N=3\} + \cdots$

$= \sum_{k=1}^{\infty} \sum_{i=1}^{k} P\{N=k\}$

$= \sum_{k=1}^{\infty} k\, P\{N=k\}$

$= E[N]$

4.10. $X \sim \text{Binomial}(n; p)$.

$E\left[\frac{1}{X+1}\right] = \sum_{k=0}^{n} \frac{1}{k+1} \cdot P\{X=k\}$

$= \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} p^k (1-p)^{n-k}$

$= \sum_{k=0}^{n} \frac{1}{k+1} \cdot \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k}$

$= \frac{1}{n+1} \sum_{k=0}^{n} \frac{n+1}{k+1} \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k}$.

$= \frac{1}{n+1} \sum_{k=0}^{n} \frac{(n+1)!}{(n-k)!\,(k+1)!} p^k (1-p)^{n-k}$.

$= \frac{1}{p(n+1)} \left[\sum_{k=0}^{n+1} \frac{(n+1)!}{(n-k)!\,((k+1))!} p^{k+1} (1-p)^{n+1-(k+1)} - (1-p)^{n+1} \right]$

$= \frac{1}{p(n+1)} \left(1 - (1-p)^{n+1}\right)$

$= \frac{1-(1-p)^{n+1}}{p(n+1)}$

4.17 (a) $P\{X \text{ is even}\} = \frac{1+(1-2p)^n}{2}$

$$= \frac{1+(1-2\cdot\frac{\lambda}{n})^n}{2}, \quad \lambda = np$$

$$\to \frac{(1+e^{-2\lambda})}{2}, \quad n\to\infty.$$

(b) $P\{X \text{ is even}\} = e^{-\lambda}\sum_n \frac{\lambda^{2n}}{(2n)!}$

$$= e^{-\lambda}\cdot \frac{e^{\lambda}+e^{-\lambda}}{2}$$

$$= \frac{1+e^{-2\lambda}}{2}$$

4.19. $EX^n = \sum_{i=0}^{\infty} i^n e^{-\lambda}\lambda^i/i!$

$$= \sum_{i=1}^{\infty} i^n e^{-\lambda}\lambda^i/i! = \sum_{i=1}^{\infty} \cancel{i}\cdot i^{n-1} e^{-\lambda}\lambda^i/\cancel{i}(i-1)!$$

$$= \sum_{i=1}^{\infty} i^{n-1} e^{-\lambda}\lambda^i/(i-1)!$$

$$= \sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda}\lambda^{j+1}/j!$$

$$= \lambda\sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda}\lambda^{j}/j!$$

$$= \lambda\cdot E[(X+1)^{n-1}]$$

$$EX^3 = \lambda E[(X+1)^2]$$

$$= \lambda\cdot\sum_{i=0}^{\infty}(i+1)^2 e^{-\lambda}\lambda^i/i!$$

$$= \lambda[\sum_{i=0}^{\infty} i^2 e^{-\lambda}\lambda^i/i! + 2\sum_{i=0}^{\infty} i e^{-\lambda}\lambda^i/i! + \sum_{i=0}^{\infty} e^{-\lambda}\lambda^i/i!]$$

$$= \lambda(E[X^2] + 2EX + 1)$$

$$= \lambda(\mathrm{Var}X + (EX)^2 + 2EX + 1)$$

$$= \lambda(\lambda + \lambda^2 + 2\lambda + 1)$$

$$= \boxed{\lambda(\lambda^2+3\lambda+1)} = \boxed{\lambda^3+3\lambda^2+\lambda}$$

Note: $E[X^3] = \lambda E[(X+1)^2]$
$= \lambda^2 E[(X+2)]$
is $\boxed{\text{not}}$ correct.
Because $(X+1)$ does not Poisson distributed!