STA532X HW7

#7.37 Let $X_1, \cdots, X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \frac{1}{2\theta}, \quad -\theta < x < \theta, \quad \theta > 0 \qquad \text{(uniform distribution)}$$

Find, if one exists, a best unbiased estimator of θ.

The joint pdf

$$L(\mathbf{x}|\theta) = \left(\frac{1}{2\theta}\right)^n \prod_{i=1} I_{(-\theta,\theta)}(x_i)$$

$$= \left(\frac{1}{2\theta}\right)^n I_{[0,\theta)}(\max_i |x_i|)$$

By the Factorization Theorem, $T = \max_i |X_i|$ is a SS.

Assume $\exists\, g$ such that $Eg(T) = 0$ for all $\theta > 0$

$T = \max_i |x_i|$ has cdf $H(t) = \left(\frac{t}{\theta}\right)^n, \quad 0 \le t < \theta$

pdf $h(t) = \frac{nt^{n-1}}{\theta^n}, \quad 0 \le t < \theta$

$$Eg(T) = \int_0^\theta g(t) \frac{nt^{n-1}}{\theta^n} dt = 0 \quad \text{for all } \theta > 0$$

implies $\int_0^\theta g(t)\, n t^{n-1} dt = 0$ for all $\theta > 0$

$g(\theta)\, n\theta^{n-1} = 0 \quad \forall \theta > 0$ by differentiating both sides and using the Fundamental Theorem.

$\Rightarrow g(t) = 0 \quad \forall t > 0$

$\Rightarrow P(g(T) = 0) = 1 \quad \forall \theta > 0$

Thus, $T = \max_i |X_i|$ is a complete SS.

$$ET = \int_0^\theta t \frac{nt^{n-1}}{\theta^n} dt = \frac{1}{\theta^n} \frac{n}{n+1} t^{n+1}\Big|_0^\theta = \frac{n\theta}{n+1}$$

$$\Rightarrow E\left(\frac{n+1}{n} T\right) = \frac{n+1}{n} ET = \frac{n+1}{n}\left(\frac{n\theta}{n+1}\right) = \theta.$$

Therefore $\frac{n+1}{n} \max_i |X_i|$ is a best unbiased estimator of θ since $\frac{n+1}{n} \max_i |X_i|$ is a function of CSS.

#7.42 Let $W_1, \cdots, W_k$ be unbiased estimators of a parameter θ with $Var(W_i) = \sigma_i^2$ and $Cov(W_i, W_j) = 0$ if $i \neq j$.

(a) Show that, of all estimators of the form $\Sigma a_i W_i$, where the a_is are constant and $E_\theta(\Sigma a_i W_i) = \theta$, the estimator $W^* = \dfrac{\Sigma W_i/\sigma_i^2}{\Sigma(1/\sigma_i^2)}$ has minimum variance.

Note:
There are several ways to solve this problem. This is one of the ways.

(soln) Let $\tau_i = \frac{1}{\sigma_i^2}$. Then we have $\underline{Var(\Sigma a_i W_i)} = \Sigma a_i^2 var(W_i) = \Sigma a_i^2 \sigma_i^2 = \underline{\dfrac{\Sigma a_i^2}{\tau_i}}$.

The condition $E_\theta(\Sigma a_i W_i) = \theta$ is equivalent $\Sigma a_i = 1$.

This problem is equivalent

to minimize $\sum_{i=1}^k \dfrac{a_i^2}{\tau_i}$ subject to $\sum_{i=1}^k a_i$.

$\Longleftrightarrow$ to minimize $\sum_{i=1}^k \dfrac{a_i^2}{\tau_i^2} \cdot \tau_i$ subject to $\sum_{i=1}^k \dfrac{a_i}{\tau_i} \cdot \tau_i$

$\Longleftrightarrow$ to minimize $\sum_{i=1}^k \dfrac{a_i^2}{\tau_i^2} \dfrac{\tau_i}{\sum_{i=1}^k \tau_i}$ subject to $\sum_{i=1}^k \dfrac{a_i}{\cancel{\tau_i}} \dfrac{\cancel{\tau_i}}{\sum_{i=1}^k \tau_i} = \dfrac{1}{\sum_{i=1}^k \tau_i}$ (by dividing by $\sum_{i=1}^k \tau_i$)

can be written as

to minimize $\sum_{i=1}^k x_i^2 P_i$ subject to $\sum_{i=1}^k x_i P_i = b$. $\qquad \dfrac{\Sigma W_i}{\tau_i}$

, where $x_i = \dfrac{a_i}{\tau_i}$, $P_i = \dfrac{\tau_i}{\Sigma \tau_i}$ and $b = \dfrac{1}{\sum_{i=1}^k \tau_i}$.

By the Lemma from Dr. Huffer's Note., the minimum variance of W^* is achieved by taking $x_i = b$. for all i or taking $\dfrac{a_i}{\tau_i} = \dfrac{1}{\Sigma \tau_i}$ or $a_i = \dfrac{\tau_i}{\Sigma \tau_i}$ ∎

<u>Lemma</u> : Let $P_1, \cdots, P_k$ satisfy $P_i \geq 0$ and $\Sigma P_i = 1$. The minimum value of $\Sigma x_i^2 P_i$ subject to the constraint $\Sigma x_i P_i = b$ is attained by setting $x_i = b$ for $i = 1, 2, \cdots, k$.

#7.42

(b) Show that $\mathrm{Var}(W^*) = \dfrac{1}{\Sigma\left(\frac{1}{\sigma_i^2}\right)}$.

Let $\tau_i = \dfrac{1}{\sigma_i^2}$.

$$W^* = \frac{\Sigma W_i/\sigma_i^2}{\Sigma(1/\sigma_i^2)} = \frac{\Sigma \tau_i W_i}{\Sigma \tau_i}$$

$$\mathrm{Var}(W^*) = \mathrm{Var}\left(\frac{\Sigma \tau_i W}{\Sigma \tau_i}\right) = \left(\frac{1}{\Sigma \tau_i}\right)^2 \sum_{i=1} \tau_i^2 \,\mathrm{Var}(W_i)$$

$$= \left(\frac{1}{\Sigma \tau_i}\right)^2 \Sigma \tau_i^2 \sigma_i^2$$

$$= \left(\frac{1}{\Sigma \tau}\right)^2 \Sigma \frac{\tau_i^{\cancel{2}}}{\cancel{\tau_i}} \qquad \left(\because \sigma_i^2 = \frac{1}{\tau_i}\right)$$

$$= \left(\frac{1}{\Sigma \tau_i}\right)^{\cancel{2}} \cancel{\left(\Sigma \tau_i\right)}$$

$$= \frac{1}{\Sigma \tau_i} = \frac{1}{\Sigma(1/\sigma_i^2)} \quad \blacksquare$$

7.46 Let X_1, X_2 and X_3 be a random sample of size three from a unifom $(\theta, 2\theta)$ distribution, where $\theta > 0$.

For $\theta>0$, pdf $\begin{cases} \frac{1}{\theta} & \text{for } x \in (\theta, 2\theta) \\ 0 & \text{o.w.} \end{cases}$

(a) Find the method of moments estimator θ.

$$EX = \frac{2\theta+\theta}{2} = \frac{3\theta}{2}.$$

Then set $\bar{X} = \frac{3\theta}{2} \Rightarrow \underline{\tilde{\theta}_{MOM} = \frac{2}{3}\bar{X}}$

(b) Find the MLE, $\hat{\theta}$, and find a constant k such that $E_\theta(k\hat{\theta}) = \theta$.

$$L(X|\theta) = \prod_{i=1}^{n} \frac{1}{2\theta-\theta} I(\theta \le x_i \le 2\theta)$$

$$= \frac{1}{\theta^n} I(x_{(1)} \ge \theta)\, I(x_{(n)} \le 2\theta)$$

$$= \frac{1}{\theta^n} I_{(x_{(1)} \ge \theta)} I_{\left(\frac{x_{(n)}}{2} \le \theta\right)} = \begin{cases} \theta^{-n} & \text{for } \theta < x_{(1)},\ X_{(n)} < 2\theta \\ 0 & \text{o.w.} \end{cases}$$

where $x_{(1)} = \min_i x_i$ and $x_{(n)} = \max_i x_i$

since $\frac{1}{\theta^n}$ is decreasing, so $L(X|\theta)$ is maximimized at $\underline{\hat{\theta} = \frac{x_{(n)}}{2}}$.

Thus $\hat{\theta} = \frac{X_{(n)}}{2}$ is MLE.

Note:
We know that if $U_1, \dots, U_n$ be iid Unifom $(0,1)$, then $U_{(k)}$ is a Beta distribution with $(k, n-k+1)$ and Expected value is

$$E(X_{(k)}) = \frac{k}{n+1}.$$

From the given, we know

$n = 3$

$X_1, \dots, X_n \sim$ Uniform $(\theta, 2\theta)$ Let $U = \frac{X}{\theta} - 1 \sim$ Unif $(0,1)$

$\theta < X_i < 2\theta$ $i=1,\dots,n.$

$0 < X - \theta < \theta$

$0 < \frac{X-\theta}{\theta} < 1$ $(\because \theta > 0)$

$U = \underbrace{\quad}$

$$E(U_{(n)}) = \frac{n}{n+1}$$

so $E\left(\frac{X_{(n)}}{\theta} - 1\right) = \frac{n}{n+1} \Rightarrow \underline{EX_{(n)} = \frac{2n+1}{n+1}\theta}$

7.46

(b) Continue..

Recall that we get $\hat{\theta} = \frac{X_{(n)}}{2}$.

$E(\hat{\theta}) = E\left(\frac{X_{(n)}}{2}\right) = \frac{1}{2}E(X_{(n)}) = \frac{2n+1}{2(n+1)}\theta$.

Thus when we choose $k = \frac{2(n+1)}{2n+1}$ such that $E(k\hat{\theta}) = \theta$.

(c) Which of the two estimators can be improved by using sufficiency? How?

$\hat{\theta}_{MLE} = \frac{X_{(n)}}{2}$ is a function of $T(x) = (X_{(1)}, X_{(n)})$, while

$\tilde{\theta}_{MOM} = \frac{2}{3}\bar{X}$ is <u>not a function</u> of $T(x) = (X_{(1)}, X_{(n)})$ because

$n > 2$. Therefore, $\tilde{\theta}_{MOM}$ estimator can be improved

by using sufficiency.

(d) Find the method of moments estimate and the MLE of θ based on the data 1.29, .86, 1.33.

$\tilde{\theta}_{MOM} = \frac{2}{3}\bar{X} = \frac{2}{3}(1.16)$

$\hat{\theta}_{MLE} = \frac{X_{(n)}}{2} = \frac{1.33}{2}$

#7.47 Suppose that when the radius of a circle is measured, an error is made that has a $n(0, \sigma^2)$ distribution.
If n independent measurements are made, find an unbiased estimator of the area of the circle. Is it best unbiased?

$$X_i \overset{ind}{\sim} n(r, \sigma^2) \quad (i = 1, \cdots, n) \qquad \hat{r} = r + e$$

$$\bar{X} \sim n\left(r, \frac{\sigma^2}{n}\right) \qquad e \sim n(0, \sigma^2)$$

$$\frac{\sigma^2}{n} = Var(\bar{X}) = E\bar{X}^2 - (E\bar{X})^2 = E\bar{X}^2 - r^2$$

$$\Rightarrow \quad E\bar{X}^2 = \frac{\sigma^2}{n} + r^2 \quad \Rightarrow \quad r^2 = E\bar{X}^2 - \frac{\sigma^2}{n}$$

Since $\bar{X}$ is CSS and σ^2 is assumed to be known, then

$$E\left(\pi\bar{X}^2 - \pi\frac{\sigma^2}{n}\right) = \pi\left(E\bar{X}^2 - \frac{\sigma^2}{n}\right) = \pi r^2 \quad \text{is} \therefore \text{ best unbiased estimator.}$$

\# 7.49 Let $X_1, X_2, \cdots, X_n$ be iid $\exp(\lambda)$

(a) Find an unbiased estimator of λ. based on $Y = \min_i X_i = X_{(1)}$.

$$f_Y(y) = \frac{n!}{(n-1)!} \frac{1}{\lambda} e^{-\frac{y}{\lambda}} \left[1-(1-e^{-y/\lambda})\right]^{n-1} = \frac{n}{\lambda} e^{-ny/\lambda}$$

$$= \frac{1}{\left(\frac{\lambda}{n}\right)} e^{-y/\left(\frac{\lambda}{n}\right)} \sim \exp\left(\frac{\lambda}{n}\right)$$

Thus, $EY = \frac{\lambda}{n} \Rightarrow E(nY) = nEY = n \cdot \frac{\lambda}{n} = \lambda$.

$\therefore$ nY is an unbiased estimator of λ.

(b) Find a better estimator than the one in part (a).

Prove that it is better.

since $X \overset{iid}{\sim} \exp(\lambda)$, so $T = \Sigma X_i$ is CSS (Complete Sufficient Statistic)

We know $T = \Sigma X_i$ is CSS. and $E(\Sigma X_i) = E(nX_1) = nE(X_1) = n\lambda$.

since $E(\Sigma X_i) = n\lambda$,

we have $(nY = nX_{(1)} \mid \Sigma X_i) = \frac{\Sigma X_i}{n}$, which is a function of $T = \Sigma X_i$ (an unbiased estimator of λ). Any function of $T = \Sigma X_i$ (CSS) is the best unbiased estimator of λ.

Therefore $\frac{\Sigma X_i}{n}$ is the best unbiased estimator of λ.

(c) From the part (a), we obtain $\hat{\lambda} = 601.2$ and from the part (b), we obtain $\hat{\lambda} = 128.8$.

This implies that the exponential model may not be a good assumption.

#7.50. Let $X_1, X_2, \dots, X_n$ be iid $\underbrace{n(\theta, \theta^2)}$ $\overset{\sigma^2}{\scriptstyle //}$, $\theta > 0$.

For this model both $\bar{X}$ and cS are unbiased estimator of θ, where

$$c = \frac{\sqrt{n-1}\,\Gamma((n-1)/2)}{\sqrt{2}\,\Gamma(n/2)}.$$

(a) Prove that for any number a the estimator $a\bar{X} + (1-a)(cS)$ is an unbiased estimator of θ

$$E\left(a\bar{x} + (1-a)(cS)\right) = aE(\bar{x}) + (1-a)E(cS)$$

$E(\bar{x}) = \theta$

$E(cS) = cE(S)$

since $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, so

$$E(S) = \sqrt{\frac{\sigma^2}{n-1}}\, E\left(\sqrt{\frac{S^2(n-1)}{\sigma^2}}\right) = \sqrt{\frac{\sigma^2}{n-1}} \int_0^\infty \sqrt{x}\, \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\, x^{\frac{n-1}{2}-1} e^{-\frac{x}{2}}\, dx$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \int_0^\infty \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\, x^{\frac{n}{2}-1} e^{-\frac{x}{2}}\, dx$$

$$= \sqrt{\frac{\sigma^2}{n-1}}\; \frac{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})} \underbrace{\int_0^\infty \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\, x^{\frac{n}{2}-1} e^{-\frac{x}{2}}\, dx}_{1}$$

$$= \sigma \sqrt{\frac{2}{n-1}}\, \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} = c^{-1}\sigma$$

Then $E(cS) = cE(S) = c(c^{-1}\sigma) = \sigma = \theta \quad (\because \sigma^2 = \theta^2)$

Therefore
$$E(a\bar{x} + (1-a)(cS)) = aE(\bar{x}) + (1-a)E(cS)$$
$$= a\theta + (1-a)\theta = \theta.$$

$\therefore$ $a\bar{x} + (1-a)cS$ is an unbiased estimator of θ.

7.50

(b) Find the value of a that produces the estimator with minimum variance.

Given $x_i \overset{iid}{\sim} n(\theta, \theta^2)$

$$Var(a\bar{x} + (1-a)cS) = a^2 var(\bar{x}) + (1-a)^2 Var(cS)$$

$$var(\bar{x}) = \frac{\theta^2}{n}$$

$$var(cS) = E(c^2S^2) - \{E(cS)\}^2$$

$$= c^2 ES^2 - \theta^2 \quad (\text{From the resuet of the part (a)}, \ E(cs) = \theta)$$

$$= c^2\theta^2 - \theta^2$$

$$= (c^2-1)\theta^2$$

$$\therefore \ Var(a\bar{x} + (1-a)cS) = a^2 \cdot \frac{\theta^2}{n} + (1-a)^2(c^2-1)\theta^2$$

$$= \theta^2\left(\frac{a^2}{n} + (1-a)^2(c^2-1)\right)$$

$$= \theta^2\left(\frac{a^2}{n} + (a^2-2a+1)(c^2-1)\right)$$

$$= \theta^2\left(\underbrace{\frac{n(c^2-1)+1}{n}}_{A} a^2 - 2\underbrace{(c^2-1)}_{B}a + \underbrace{(c^2-1)}_{C}\right)$$

We know the quadratic function is minimized at $a = \frac{B}{A} = \frac{c^2-1}{(c^2-1)+\frac{1}{n}}$.

Thus $a = \frac{c^2-1}{(c^2-1)+\frac{1}{n}}$.

7.50

(c) Show that $(\bar{X}, S^2)$ is a sufficient statistic for θ but it is not a complete sufficient statistic.

From Example 6.2.9 (Casella & Berger), we know

$(\bar{X}, S^2)$ is a SS for (μ, σ^2).

Assume $\exists$ a function $f(\bar{X}, S) = \bar{X} - cS$.

$$E(f(\bar{x}, s) | \theta) = E(\bar{x} - cs) = E(\bar{X}) - E(cs) = \theta - \theta = 0$$

But $f(\bar{X}, S) = \bar{X} - cS \neq 0$

Thus $(\bar{X}, S)$ is not complete.

#7.59 Let $X_1, X_2, \cdots, X_n$ be iid $N(\mu, \sigma^2)$. Find the best unbiased estimator of σ^p, where p is a known positive constant, not necessarily an integer.

We know $T = \dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. The pdf of χ^2_{n-1} is $f_T(t) = \dfrac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})} t^{\frac{n-1}{2}-1} e^{-\frac{t}{2}}$

$$E\,T^{p/2} = \int_0^\infty t^{p/2} f_T(t)\,dt = \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\int_0^\infty t^{\frac{p+n-1}{2}-1} e^{-\frac{t}{2}}\,dt$$

$$= \frac{2^{\frac{p+n-1}{2}}\Gamma(\frac{p+n-1}{2})}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})} \underbrace{\int_0^\infty \frac{1}{2^{\frac{p+n-1}{2}}\Gamma(\frac{p+n-1}{2})} t^{\frac{p+n-1}{2}-1} e^{-\frac{t}{2}}\,dt}_{1}$$

$$= \frac{2^{\frac{p}{2}}\Gamma(\frac{p+n-1}{2})}{\Gamma(\frac{n-1}{2})} = C(p,n)$$

a constant

Thus, $E\left(\dfrac{(n-1)S^2}{\sigma^2}\right)^{p/2} = C(p,n)$

$$(n-1)^{p/2} S^p E(\sigma^{-2}) = C(p,n)$$

$\dfrac{(n-1)^{p/2}S^p}{C(p,n)} = E(\sigma^2)$. Thus $\dfrac{(n-1)^{p/2}S^p}{C(p,n)}$ is unbiased estimator of σ^p.

But Theorem 6.2.25, $(\bar{X}, S^2)$ is a complete SS.

Thus, the unbiased $\dfrac{(n-1)^{p/2}S^p}{C(p,n)}$ is a function of $(\bar{X}, S^2)$.

Therefore, $\dfrac{(n-1)^{p/2}S^p}{C(p,n)}$ is the best unbiased estimator.

#7.60 Let $X_1, \cdots, X_n$ be iid gamma (α, β) with α known.
Find the best unbiased estimator of $1/\beta$.

$$L(\alpha,\beta) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1} e^{-x_i/\beta}}{\beta^\alpha \Gamma(\alpha)} = \frac{\prod_{i=1}^{n} x_i^{\alpha-1} e^{-\Sigma x_i/\beta}}{\beta^{n\alpha} (\Gamma(\alpha))^n} = \frac{1}{\beta^{n\alpha}(\Gamma(\alpha))^n} \overbrace{\left(\prod_{i=1}^{n} x_i^{\alpha-1}\right)}^{T_1} \exp\Big(-\overbrace{\frac{\Sigma x_i}{\beta}}^{T_2}\Big)$$

By Factorization Theorem.

Take a log.

$$\ell(\alpha,\beta) = (\alpha-1)\sum_{i=1}^{n} \log x_i - \frac{\Sigma x_i}{\beta} - n\alpha \log\beta - n\log\Gamma(\alpha).$$

Since α is known, now it's a 1pef. Thus, $T = \Sigma x_i$ is the natural SS for β.

We know that $T = \Sigma x_i$ is CSS and $T \sim \text{Gamma}(n\alpha, \beta)$.

$$ET^k = \int \frac{t^{n\alpha+k-1}}{\beta^{n\alpha}\Gamma(n\alpha)} e^{-t/\beta}\, dt$$

$$= \frac{\beta^{(n\alpha+k)}\Gamma(n\alpha+k)}{\beta^{n\alpha}\Gamma(n\alpha)} \underbrace{\int \frac{t^{(n\alpha+k)-1}}{\beta^{(n\alpha+k)}\Gamma(n\alpha+k)} e^{-t/\beta}\, dt}_{1}$$

Note
$\Gamma(n) = (n-1)!$

$$= \frac{\beta^k \Gamma(n\alpha+k)}{\Gamma(n\alpha)} \quad (\text{for } n\alpha+k>0 \text{ or } k>-n\alpha.)$$

Set $k=-1$. Then we get

$$= \frac{\beta^k (n\alpha-2)!}{(n\alpha-1)!} = \frac{\beta^{-1}}{(n\alpha-1)} = \frac{1}{\beta(n\alpha-1)} \quad (n\alpha>1).$$

$$ET^{-1} = E\frac{1}{T} = \frac{1}{\beta(n\alpha-1)} \quad (n\alpha>1).$$

$E\frac{(n\alpha-1)}{T} = \frac{1}{\beta}$. Thus $\frac{n\alpha-1}{T}$ is an unbiased estimator of $1/\beta$. It's best unbiased because T is a CSS and $\frac{(n\alpha-1)}{T}$ is a function of a CSS!