Homework 7 Solutions

October 24, 2013 STA 4442/5440

Solutions of problems from Sheldon Ross book (8th Edition) Chapter 5

5.6. a) Apply integration by parts twice to obtain,

$$E[X] = \frac{1}{4}\int_0^\infty x^2 e^{-x/2} dx = 4$$

b) By symmetry of $f(x)$ about $x = 0$, $E[X] = 0$.

c) $E[X] = \int_5^\infty \frac{5}{x} dx = \infty$

5.7. $\int_0^1 (a + bx^2)dx = 1$ or, $a + b/3 = 1$. $\int_0^1 x(a + bx^2)dx = 1$ or, $a/2 + b/4 = 3/5$. Solving for a and b, we get $a = 3/5$ and $b = 6/5$.

5.9. If s units are stocked and the demand is X, then the profit, $P(s)$, is given by

$$P(s) = \begin{cases} bX - (s - X) if X \leq s \\ bs\, if X > s \end{cases}$$

Hence

$$\begin{aligned} E(P(s)) &= \int_0^s (bx - (s - x))f(x)dx + \int_s^\infty sbf(x)dx \\ &= (b + l)\int_0^s xf(x)dx - sl\int_0^s f(x)dx + sb\left\{1 - \int_0^s f(x)dx\right\} \\ &= sb + (b + l)\int_0^s (x - s)f(x)dx \end{aligned}$$

Differentiation yields using the Leibnitz rule

$$\begin{aligned} \frac{d}{ds}E(P(s)) &= b + (b + l)\frac{d}{ds}\left\{\int_0^s xf(x)dx - s\int_0^s f(x)dx\right\} \\ &= b - (b + l)\int_0^s f(x)dx \end{aligned}$$

Equating to zero shows that the maximal expected profit is obtained when s is chosen so that

$$F(s) = \frac{b}{b + l}$$

5.11. Let U denote the random point picked up from the interval $[0, L]$. Then $U \sim \text{Unif}([0, L])$. The density of U is given by

$$f_U(x) = \begin{cases} 1/L \,\text{if}\, x \in [0, L] \\ 0, \; o/w \end{cases}$$

Define X to be another random variable denoting the ratio of the shorter interval to the longer interval. Then

$$X = \begin{cases} \frac{U}{L-U}, \,\text{if}\, U < 0.5L \\ \frac{L-U}{U}, \,\text{if}\, U \geq 0.5L \end{cases}$$

We need to find $P(X < 0.25)$. Note that

$$\begin{aligned} P(X < 0.25) &= P(\{X < 0.25\} \cap \{U < 0.5L\}) + P(\{X < 0.25\} \cap \{U \geq 0.5L\}) \\ &= P\left(\left\{\frac{U}{L-U} < 0.25\right\} \cap \{U < 0.5L\}\right) + P\left(\left\{\frac{L-U}{U} < 0.25\right\} \cap \{U \geq 0.5L\}\right) \\ &= P\Big(U < L/5\Big) + P\left(\Big\{U > 4L/5\Big\} \cap \{U \geq 0.5L\}\right) \\ &= P\Big(U < L/5\}\Big) + P\Big(\{U \geq 4L/5\}\Big) \\ &= 2/5 \end{aligned}$$

5.12. Let X denote the location of the breakdown which is uniformly distributed in $(0, 100)$. In the first case, the towing distance will be

$$Y = \begin{cases} X, X \leq 25 \\ 50 - X, X \in (25, 50] \\ X - 50, X \in (50, 75] \\ 100 - X, X \in (75, 100] \end{cases}$$

and in the second scenario the towing distance will be

$$Z = \begin{cases} 25 - X, X \leq 25 \\ X - 25, X \in (25, 37.5] \\ 50 - X, X \in (37.5, 50] \\ X - 50, X \in (50, 62.5] \\ 75 - X, X \in (62.5, 75] \\ X - 75, X \in (75, 100) \end{cases}$$

$E(Y) = 12.5$ in the first cases whereas $E(Z) = 9.375$ in the second case and hence the second case will be more efficient.

Theoretical Exercises

5.2.

$$\int_0^\infty P(Y < -y)dy = \int_0^\infty \int_{-\infty}^{-y} f_Y(x)dxdy = \int_{-\infty}^0 \int_0^{-x} f_Y(x)dydx = -\int_{-\infty}^0 xf_Y(x)dx$$

Similarly,

$$\int_0^\infty P(Y > y)dy = \int_0^\infty xf_Y(x)dx$$

5.7. $S.D(aX + b) = Var(aX + b) = \sqrt{a^2\sigma^2} = |a|\sigma$