#7.38 Let $X_1, \dots, X_n$ be a r.v.

Is there a function of θ, say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the C-R Lower Bound?

(a) $f(x|\theta) = \theta x^{\theta-1}$, $0<x<1$, $\underline{\underline{\theta>0}}$.

$$L(\theta|X) = \prod_{i=1}^{n} \theta x_i^{\theta-1} I_{(0,1)}(x_i)$$

$$= \theta^n \cdot \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$

$$\log L(\theta|x) = l(\theta|x) = n\log\theta + (\theta-1)\Sigma \log x_i = n\log\theta + \theta\Sigma\log x_i - \Sigma\log x_i$$

$$\frac{\partial l(\theta|x)}{\partial\theta} = \frac{n}{\theta} + \Sigma\log x_i = n\left[\frac{1}{\theta} + \frac{\Sigma\log x_i}{n}\right] = -n\left[\underbrace{-\frac{\Sigma\log x_i}{n}}_{W(X)} - \underbrace{\frac{1}{\theta}}_{\tau(\theta)}\right].$$

Since $-\frac{\Sigma\log x_i}{n}$ is an unbiased estimator of $\frac{1}{\theta}$,

so $-\frac{\Sigma\log x_i}{n}$ attains the C-R Lower Bound by Corollary 7.3.15 (Attainment)

Also since $-\frac{\Sigma\log x_i}{n}$ is a function of the CSS $T = \Sigma\log x_i$, so it is the UMVUE of $\frac{1}{\theta}$.

(b) $f(x|\theta) = \frac{\log(\theta)}{\theta-1}\theta^x$, $0<x<1$, $\underline{\underline{\theta>1}}$

$$L(\theta|x) = \prod_{i=1}^{n} \frac{\log(\theta)}{\theta-1}\theta^{x_i} I_{(0,1)}(x_i)$$

$$l(\theta|x) = \log L(\theta|x) = \Sigma\log\log\theta - \Sigma\log(\theta-1) + \Sigma x_i\log\theta$$

$$\frac{\partial l(\theta|x)}{\partial\theta} = n\left(\frac{1}{\theta\log\theta} - \frac{1}{\theta-1}\right) + \frac{\Sigma x_i}{\theta} = \frac{n}{\theta}\left[\left(\frac{1}{\log\theta} - \frac{\theta}{\theta-1}\right) + \bar{x}\right] = \frac{n}{\theta}\left[\underbrace{\bar{x}}_{W(X)} - \underbrace{\left(\frac{\theta}{\theta-1} - \frac{1}{\log\theta}\right)}_{\tau(\theta)}\right]$$

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We know $T=\Sigma X_i$ is CSS. Since $\bar{X}$, which is an unbiased estimator of $\left(\frac{\theta}{\theta-1}-\frac{1}{\log\theta}\right)$ and a function of CSS $T=\Sigma X_i$, $\bar{X}$ is the UMVUE. Also $\bar{X}$ attains the C-R lower bound by Corollary 7.3.15 (Attainment).

#7.40 Let $X_1, \cdots, X_n \overset{iid}{\sim}$ Bernoulli(p).
Show that the variance of $\bar{X}$ attains the C-R Lower Bound, and hence $\bar{X}$ is the best unbiased estimator of p.

$$l(x|p) = \log L(x|p) = \log \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$= \Sigma x_i \log p + \Sigma(1-x_i)\log(1-p)$$

$$\frac{\partial l(x|p)}{\partial p} = \underbrace{\Sigma\frac{x_i}{p} - \Sigma\frac{(1-x_i)}{1-p}} = \frac{\Sigma x_i - np}{p(1-p)} = \frac{n(\bar{x}-p)}{p(1-p)} = \frac{n}{p(1-p)}[\underbrace{\bar{x}}_{W(x)} - \underbrace{p}_{\tau(\theta)}]$$

$\bar{x}$ is the UMVUE of p and attains the C-R Lower Bound by Corollary 7.3.15 (Attainment). Alternatively, we can use $\frac{\tau'(\theta)}{-nE_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right)}$.

$$-nE\left(\frac{\partial^2 l(x|p)}{\partial p^2}\right) = -nE\left(\frac{\partial^2}{\partial p^2}\log p^x(1-p)^{1-x}\right) = -nE\left[\frac{\partial^2}{\partial p^2}\left(x\log p + (1-x)\log(1-p)\right)\right]$$

$$= -nE\left[\frac{\partial}{\partial p}\left(\frac{x}{p} - \frac{1-x}{1-p}\right)\right] = -nE\left(-\frac{x}{p^2} - \frac{1-x}{(1-p)^2}\right)$$

$$\left(\begin{array}{c}\text{Since } x \overset{iid}{\sim} Ber(p) \\ E(x)=p\end{array}\right)$$

$$= +\frac{n}{p^2}E(x) + \frac{n}{(1-p)^2}(1-E(x)) = \frac{n}{p} + \frac{n}{1-p} = \frac{n}{p(1-p)}.$$

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Then we have $\dfrac{\tau'(\theta)}{-nE_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right)} = \dfrac{1}{\frac{n}{p(1-p)}} = \dfrac{p(1-p)}{n} = Var(\bar{X}).$

$\left(\because \tau'(\theta) = \frac{\partial}{\partial p} p = 1 \right)$

and $E(\bar{X}) = p.$

Therefore, $\bar{X}$ attains the Cramér-Rao Bound.

#7.52. Let $X_1, \dots, X_n \overset{iid}{\sim}$ Poisson(λ).

(a) Prove that $\bar{X}$ is the best unbiased estimator of λ w\o using the Cramér-Rao Theorem.

Since Poisson(λ) is a exponential family,

we know that $T = \Sigma X_i$ is CSS.

$\bar{X}$ is a function of $T = \Sigma X_i$ that is an unbiased estimator of λ and it is the unique best unbiased estimator of λ.

Given $X_i \overset{iid}{\sim}$ Poisson(λ),

then $E(X_i) = \lambda$

and $E(\Sigma X_i) = n\lambda.$

$E(\bar{X}) = E\left(\frac{\Sigma X_i}{n}\right) = \lambda.$

Thus, $\bar{X}$ is the best unbiased estimator of λ as we desired.

#7.52

(b) Prove the rather remarkable identity $E(S^2|\bar{X}) = \bar{X}$, and use it to explicitly demonstrate that $\text{Var}\, S^2 > \text{Var}\, \bar{X}$.

We know S^2 is unbiased such that $ES^2 = \lambda$.

$\bar{X}$ is 1-1 function of ΣX_i so $\bar{X}$ is a CSS as well.

$\therefore$ $E(S^2|\bar{X})$ is an unbiased estimator of λ. $(E(S^2|\bar{X}) = E(S^2) = \bar{X})$

By Theorem 7.3.23, it is also the unique best unbiased estimator of λ.

$$\text{Var}\, S^2 = \text{Var}\left(E(S^2|\bar{X})\right) + E\left(\text{Var}(S^2|\bar{X})\right)$$

$$= \text{Var}(\bar{X}) + E\left(\text{Var}(S^2|\bar{X})\right)$$

Therefore $\text{Var}\, S^2 > \text{Var}\, \bar{X}$.

(c) Using completeness, can a general theorem be formulated for which the identity in part (b) is a special case?

Let $T(x) = \Sigma X_i$ be a CSS and let $T^c(x)$ be any statistic other than $T(x)$ such that $ET(x) = ET^c(x)$.

Then $E\left[T^c(x) | T(x)\right] = T(x)$ and $\text{Var}\, T^c(x) > \text{Var}\, T(x)$.