Homework 9 Solutions

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6.24 (a)$P(N = n) = p_0^{n-1}(1 - p_0)$
(b) $P(X = j) = \sum_{n=1}^{\infty} P(N = n, X = j) = \sum_{n=1}^{\infty} p_0^{n-1} p_j = \frac{p_j}{(1-p_0)}$
(c)$P(N = n, X = j) = p_0^{n-1} p_j = P(N = n)P(X = j)$
(d) and (e) It might seem intuitively that X is independent of N, but N is not independent of X (since you may think that the occurrence of X influences on the occurrence of N. But this is not true as it follows from (c) that X and N are independent.

1. Let X and Y denotes the number of males and females that enter the post office. Then

$$\begin{aligned} P(X = i, Y = j) &= P(X = i, Y = j \mid X + Y = i + j)P(X + Y = i + j) + \\ &\quad P(X = i, Y = j \mid X + Y \neq i + j)P(X + Y \neq i + j) \\ &= P(X = i, Y = j \mid X + Y = i + j)P(X + Y = i + j) \end{aligned}$$

Since $(X + Y) \sim Poiss(\lambda), P(X + Y = i + j) = \exp\{-\lambda\}\lambda^{i+j}/(i + j)!$ and

$$P(X = i, Y = j \mid X + Y = i + j) = \binom{i+j}{i} p^i (1 - p)^j$$

Hence

$$\begin{aligned} P(X = i, Y = j) = \binom{i+j}{i} p^i (1 - p)^j \exp\{-\lambda\}\lambda^{i+j}/(i + j)! \\ = e^{-\lambda p} \frac{(\lambda p)^i}{i!} e^{-\lambda(1-p)} \frac{(\lambda(1 - p))^j}{j!} \end{aligned}$$

Since the joint density of X and Y factors into 2 parts involving i and j separately, we have X and Y independent with $X \sim Poisson(\lambda p)$ and $Y \sim Poisson(\lambda(1 - p))$

6.1. (a) $P\{X=1, Y=2\} = P(\text{Die1}=1, \text{Die2}=1) = \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{1}{36}}$

$P\{X=2, Y=3\} = P\{\text{Die1}=1, \text{Die2}=2\} + P\{\text{Die1}=2, \text{Die2}=1\} = \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{2}{36}}$

$P\{X=2, Y=4\} = P\{D1=2, D2=2\} = \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{1}{36}}$

$P\{X=3, Y=4\} = P\{D1=3, D_2=1\} + P\{D1=1, D2=3\} = \boxed{\frac{2}{36}}$

$P\{X=3, Y=5\} = \boxed{\frac{2}{36}}$

D1	D2
3	2
2	3

$P\{X=3, Y=6\} = \boxed{\frac{1}{36}}$

D1	D2
3	3

$P\{X=4, Y=5\} = \boxed{\frac{2}{36}}$

D_1	D_2
4	1
1	4

$P\{X=4, Y=6\} = \boxed{\frac{2}{36}}$

D1	D2
4	2
2	4

$P\{X=4, Y=7\} = \boxed{\frac{2}{36}}$

D1	D2
4	3
3	4

$P\{X=4, Y=8\} = \boxed{\frac{1}{36}}$

D_1	D_2
4	4

$P\{X=5, Y=6\} = \boxed{\frac{2}{36}}$

D_1	D_2
5	1
1	5

$P\{X=5, Y=7\} = \boxed{\frac{2}{36}}$

$P\{X=5, Y=8\} = P\{X=5, Y=9\} = \boxed{\frac{2}{36}}$

$P\{X=5, Y=10\} = \boxed{\frac{1}{36}}$

$P\{X=6, Y=7\} = P\{X=6, Y=8\} = P\{X=6, Y=9\} = P\{X=6, Y=10\} = P\{X=6, Y=11\} = \boxed{\frac{2}{36}}$

$P\{X=6, Y=12\} = \boxed{\frac{1}{36}}$

(b). $P(X=1, Y=1) = \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{1}{36}}$ $\quad$ D_1 1, D_2 1

$P\{X=1, Y=2\} = \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{1}{36}}$ $\quad$ D_1 1, D_2 2

Similarly,

$P\{X=i, Y=j\} = \frac{1}{6}\cdot\frac{1}{6}$, $\quad$ D_1 i, D_2 j

In conclusion,

$P\{X=i, Y=j\} = \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{1}{36}}$, $\underbrace{i=1,2,\cdots,6,\ j=1,2,\cdots,6}_{\text{Don't forget to write the domain.}}$

(C) $P\{X=1, Y=1\} = \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{1}{36}}$ $\quad$ D_1 1, D_2 1

$P\{X=1, Y=2\} = \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{1}{36}}$ $\quad$ D_1 1, D_2 1

$P\{X=1, Y=3\} = \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{1}{36}}$ $\quad$ D_1 1, D_2 3

Similarly

$P\{X=i, Y=j\} = \frac{1}{6}\cdot\frac{1}{6}$. $\quad$ D_1 i, D_2 j

$P\{X=i, Y=j\} = \frac{1}{6}\cdot\frac{1}{6} = \boxed{\frac{1}{36}}$, $i=1,2,\cdots,6$, $j=1,2,\cdots,6$

6.2 (a) ~~Each ball has $\frac{1}{5+8} = \frac{1}{13}$ probability to be selected.~~

$P(X_1=0, X_2=0) = \frac{8}{13}\cdot\frac{8-1}{13-1} = \frac{8}{13}\cdot\frac{7}{12} = \boxed{\frac{14}{39}} = \boxed{0.36}$

$P(X_1=0, X_2=1) = \frac{8}{13}\cdot\frac{5}{13-1} = \frac{8}{13}\cdot\frac{5}{12} = \boxed{\frac{10}{39}} = \boxed{0.26}$

$P(X_1=1, X_2=0) = \frac{5}{13}\cdot\frac{8}{13-1} = \frac{5}{13}\cdot\frac{8}{12} = \boxed{\frac{10}{39}} = \boxed{0.26}$

$P(X_1=1, X_2=1) = \frac{5}{13}\cdot\frac{5-1}{13-1} = \boxed{\frac{5}{39}} = \boxed{0.13}$

(b)

X_1	$X_2=0, X_3=0$	$X_2=0, X_3=1$	$X_2=1, X_3=0$	$X_2=1, X_3=1$
0	$\frac{8}{13}\cdot\frac{8-1}{13-1}\cdot\frac{8-2}{13-2} = \boxed{\frac{28}{143}} = \boxed{0.20}$	$\frac{8}{13}\cdot\frac{8-1}{13-1}\cdot\frac{5}{13-2} = \boxed{\frac{70}{429}} = \boxed{0.16}$	$\frac{8}{13}\cdot\frac{5}{13-1}\cdot\frac{8-1}{13-2} = \boxed{\frac{70}{429}} = \boxed{0.16}$	$\frac{8}{13}\cdot\frac{5}{13-1}\cdot\frac{5-1}{13-2} = \boxed{\frac{40}{429}} = \boxed{0.09}$
1	$\frac{5}{13}\cdot\frac{8}{13-1}\cdot\frac{8-1}{13-2} = \boxed{\frac{70}{429}} = \boxed{0.16}$	$\frac{5}{13}\cdot\frac{8}{12}\cdot\frac{\cancel{8}\,5-1}{11} = \boxed{\frac{40}{429}} = \boxed{0.09}$	$\frac{5}{13}\cdot\frac{5-1}{12}\cdot\frac{8}{11} = \boxed{\frac{40}{429}} = \boxed{0.09}$	$\frac{5}{13}\cdot\frac{5-1}{12}\cdot\frac{5-2}{11} = \boxed{\frac{5}{143}} = \boxed{0.03}$

6.3 (a)

~~X_2~~ Y_2 \ ~~X_1~~ Y_1	0	1
0	$\frac{12}{13}\cdot\frac{12-1}{13-1} = \boxed{\frac{132}{156}} = \boxed{0.85}$	$\frac{1}{13}\cdot\frac{11}{12} = \boxed{\frac{11}{156}} = \boxed{0.07}$
1	$\frac{12}{13}\cdot\frac{1}{12} = \boxed{\frac{1}{13}} = \boxed{0.08}$	$\frac{1}{13}\cdot\frac{1}{12} = \boxed{\frac{1}{156}} = \boxed{0.006}$

6.3 (b)

$Y_1 \downarrow$	$Y_2=0, Y_3=0$	$Y_2=0, Y_3=1$	$Y_2=1, Y_3=0$	$Y_2=1, Y_3=1$
0	$\frac{12}{13}\cdot\frac{11}{12}\cdot\frac{10}{11}=\boxed{\frac{10}{13}}$ $=\boxed{0.77}$	$\frac{12}{13}\cdot\frac{11}{12}\cdot\frac{1}{11}=\boxed{\frac{1}{13}}$ $=\boxed{0.08}$	$\frac{12}{13}\cdot\frac{1}{12}\cdot\frac{10}{11}=\boxed{\frac{10}{143}}$ $=\boxed{0.07}$	$\frac{12}{13}\cdot\frac{1}{12}\cdot\frac{1}{11}=\boxed{\frac{1}{143}}$ $=\boxed{0.007}$
1	$\frac{1}{13}\cdot\frac{11}{12}\cdot\frac{10}{11}=\boxed{\frac{5}{78}}$ $=\boxed{0.06}$	$\frac{1}{13}\cdot\frac{11}{12}\cdot\frac{1}{11}=\boxed{\frac{1}{156}}$ $=\boxed{0.006}$	$\frac{1}{13}\cdot\frac{1}{12}\cdot\frac{10}{11}=\boxed{\frac{10}{1{,}716}}$ $=\boxed{0.006}$	$\frac{1}{13}\cdot\frac{1}{12}\cdot\frac{1}{11}=\boxed{\frac{1}{1{,}716}}$ $=\boxed{0.0006}$

6.7. $P\{x_1, x_2\} = P\{X_1 = x_1, X_2 = x_2\}$

$$= \binom{n}{x_1} \cdot p^{n-x_1} \cdot (1-p)^{x_1} \cdot \binom{m}{x_2} p^{m-x_2} \cdot (1-p)^{x_2}$$

$$= \binom{x_1+1}{x_1} p (1-p)^{x_1} \binom{x_2+1}{x_2} p (1-p)^{x_2}$$

$$= (x_1+1) p (1-p)^{x_1} (x_2+1) p (1-p)^{x_2}$$

$$= (x_1+1)(x_2+1) p^2 (1-p)^{x_1+x_2}$$