October 29, 2013 Debdeep Pati

1 Jointly distributed Random variables

1. Sometimes more than one random variables are needed to study a problem.
2. Complex problems often contain more than one random variables.
3. For some problems, we may want to introduce new random variables to solve them.
4. Much of what we have learned can be readily extended to multiple random variable case. Some new concepts arise, such as Independence among random variables, Covariance, Correlations

1.1 Examples

Grade data (frequency distribution)for 200 college students taking chemistry and physics.

Table 1: default

	A=4	B=3	C=2	D = 1	Total
A=4	32	22	8	1	63
B=3	11	35	25	1	72
C=2	3	25	20	2	50
D=1	0	5	5	5	15
Total	46	87	58	9	200

Grade data (probability distribution)for 200 college students taking chemistry and physics.

1. Suppose the transcript of a student is drawn at random.
2. Find the probability that the student received a B in chemistry and a B in physics.
3. $P(X = B, Y = B) = 0.175$
4. What is the probability that the student received a grade better than B in chemistry and a grade worse than B in physics?

Table 2: default

	A=4	B=3	C=2	D = 1	Total
A=4	0.160	0.110	0.040	0.005	0.315
B=3	0.055	0.175	0.125	0.005	0.360
C=2	0.015	0.125	0.100	0.010	0.250
D=1	0.000	0.025	0.025	0.025	0.075
Total	0.230	0.435	0.290	0.045	1.00

5. $P(X > B, Y < B) = .045$

1.2 Joint Probability distributions

The joint cdf for discrete random variables is given by

$$F(a,b) = P(X \leq a, Y \leq b) = \sum_{x \leq a} \sum_{y \leq b} p(x,y)$$

Joint probability mass function: $p(x,y) = P(X = x, Y = y)$ The marginal probability mass functions for X and Y

$$P_X(x) = \sum_{y:p(x,y)>0} p(x,y)$$
$$P_Y(y) = \sum_{x:p(x,y)>0} p(x,y)$$

1.3 Example

Suppose that 15 percent of the families in a certain community have no children, 20 percent have 1, 35 percent have 2, and 30 percent have 3; and suppose, further, that in each family, each child is equally likely to be a boy or a girl. If a family is chosen at random from this community, let B be the number of boys, and G, the number of girls, in this family. Find the joint probability mass function of B and G. Possible values of B and G: 0, 1, 2, 3.
P(B = 0, G = 0) = P(no children) = .15
P(B = 0, G = 1) = P(1 girl and total of 1 child) = P(1 child)P(1 girl | 1child) = (.20)(1/2)
P(B = 0, G = 2) = P(2 girls and total of 2 children) = P(2 children)P(2 girls | 2 children) = $(.35)(1/2)^2$

Here is the joint probability table.

Table 3: default

	0	1	2	3	Total
0	0.15	0.10	0.0875	0.0375	0.375
1	0.10	0.175	0.1125	0	0.3875
2	0.0875	0.1125	0	0	0.200
3	0.0375	0	0	0	0.0375
Total	0.0375	0.3875	0.200	0.0375	1

1.4 The Multinomial distribution

Suppose that a fair die is rolled 9 times. What is the probability that 1 appears three times, 2 and 3 twice each, 4 and 5 once each, and 6 not at all?

$$\frac{9!}{3!2!2!1!1!0!}\frac{1}{6}^3\frac{1}{6}^2\frac{1}{6}^2\frac{1}{6}^1\frac{1}{6}^1$$

A sequence of n independent and identical experiments is performed. Suppose that each experiment can result in any one of r possible outcomes, with respective probabilities $p_1, p_2, \ldots, p_r, \sum_{i=1}^{r} p_i = 1$. If we let X_i denote the number of the n experiments that result in outcome number i, then

$$P(X_1 = n_1, X_2 = n_2, \ldots, X_r = n_r) = \frac{n!}{n_1!n_2!\ldots n_r!}p_1^{n_1}p_2^{n_2}\cdots p_r^{n_r}$$

where $\sum_{i=1}^{r} n_i = n$.

2 Independent Random Variables

Random variables X and Y are said to be independent if for two sets A and B of real numbers

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

For discrete random variables

$$p(x, y) = p_X(x)p_Y(y)$$

and for continuous random variables

$$f(x, y) = f_X(x)f_Y(y)$$

2.1 Examples

1. Suppose that $n + m$ independent trials, having a common success probability p, are performed. If X is the number of successes in the first n trials, and Y is the number of successes in the final m trials, then X and Y are independent, since knowing the number of successes in the first n trials does not affect the distribution of the number of successes in the final m trials (by the assumption of independent trials).