1 Jointly distributed Random variables

1. Sometimes more than one random variables are needed to study a problem.
2. Complex problems often contain more than one random variables.
3. For some problems, we may want to introduce new random variables to solve them.
4. Much of what we have learned can be readily extended to multiple random variable case. Some new concepts arise, such as Independence among random variables, Covariance, Correlations

1.1 Examples

Grade data (frequency distribution)for 200 college students taking chemistry and physics.

Table 1: default

	A=4	B=3	C=2	D = 1	Total
A=4	32	22	8	1	63
B=3	11	35	25	1	72
C=2	3	25	20	2	50
D=1	0	5	5	5	15
Total	46	87	58	9	200

Grade data (probability distribution)for 200 college students taking chemistry and physics.

1. Suppose the transcript of a student is drawn at random.
2. Find the probability that the student received a B in chemistry and a B in physics.
3. $P(X = B, Y = B) = 0.175$
4. What is the probability that the student received a grade better than B in chemistry and a grade worse than B in physics?

Table 2: default

	A=4	B=3	C=2	D = 1	Total
A=4	0.160	0.110	0.040	0.005	0.315
B=3	0.055	0.175	0.125	0.005	0.360
C=2	0.015	0.125	0.100	0.010	0.250
D=1	0.000	0.025	0.025	0.025	0.075
Total	0.230	0.435	0.290	0.045	1.00

5. $P(X > B, Y < B) = .045$

1.2 Joint Probability distributions

The joint cdf for discrete random variables is given by

$$F(a, b) = P(X \leq a, Y \leq b) = \sum_{x \leq a} \sum_{y \leq b} p(x, y)$$

Joint probability mass function: $p(x, y) = P(X = x, Y = y)$ The marginal probability mass functions for X and Y

$$P_X(x) = \sum_{y:p(x,y)>0} p(x, y)$$
$$P_Y(y) = \sum_{x:p(x,y)>0} p(x, y)$$

1.3 Example

Suppose that 15 percent of the families in a certain community have no children, 20 percent have 1, 35 percent have 2, and 30 percent have 3; and suppose, further, that in each family, each child is equally likely to be a boy or a girl. If a family is chosen at random from this community, let B be the number of boys, and G, the number of girls, in this family. Find the joint probability mass function of B and G. Possible values of B and G: 0, 1, 2, 3.
P(B = 0, G = 0) = P(no children) = .15
P(B = 0, G = 1) = P(1 girl and total of 1 child) = P(1 child)P(1 girl | 1child) = (.20)(1/2)
P(B = 0, G = 2) = P(2 girls and total of 2 children) = P(2 children)P(2 girls | 2 children) = (.35)(1/2)2

Here is the joint probability table.

Table 3: default

	0	1	2	3	Total
0	0.15	0.10	0.0875	0.0375	0.375
1	0.10	0.175	0.1125	0	0.3875
2	0.0875	0.1125	0	0	0.200
3	0.0375	0	0	0	0.0375
Total	0.0375	0.3875	0.200	0.0375	1

2 Jointly continuous random variable

If X and Y are continuous, $f(x,y)$ is called joint probability density function of X and Y if for any subset $C \subset \mathbb{R}^2$,

$$P((X,Y) \in C) = \int\int_{(x,y)\in C} f(x,y)dxdy$$

1. Double Integral: $P(X \in A, Y \in B) = \int_A \int_B f(x,y)dxdy$
2. Distribution function: $F(a,b) = P(X \in (-\infty, a], Y \in (-\infty, b])) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)dxdy$
3. By differentiation $\frac{\partial^2}{\partial a \partial b} F(a,b) = f(a,b)$
4. Marginal distribution: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy, f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$

2.1 Problem

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, \; 0 < x < \infty, 0 < y < \infty \\ 0, \; o/w \end{cases}$$

Find $P(X > 1, Y < 1), P(X < Y), P(X < a)$.

$$\begin{aligned} P(X > 1, Y < 1) &= 2\int_{x=1}^{\infty} \int_{y=0}^{1} e^{-x}e^{-2y}dydx \\ &= \int_{x=1}^{\infty} e^{-x}(1-e^{-2})dx \\ &= e^{-1}(1-e^{-2}) \end{aligned}$$

$$
\begin{aligned}
P(X < Y) &= 2\int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{-x}e^{-2y}dydx \\
&= \int_{x=0}^{\infty} e^{-x}(1-e^{-2})dx \\
&= e^{-1}(1-e^{-2})
\end{aligned}
$$

$$
\begin{aligned}
P(X < a) &= 2\int_{x=0}^{a}\int_{y=0}^{\infty} e^{-x}e^{-2y}dydx \\
&= \int_{x=0}^{a} e^{-x}dx \\
&= 1-e^{-a}
\end{aligned}
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