1 Jointly distributed random variables

1.1 Problem

Consider a circle of radius R and suppose that a point within the circle is randomly chosen in such a manner that all regions within the circle of equal area are equally likely to contain the point. (In other words, the point is uniformly distributed within the circle.) If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen, it follows, since (X, Y) is equally likely to be near each point in the circle, that the joint density function of X and Y is given by

$$f(x,y) = \begin{cases} c, \; if \; (x^2+y^2) \leq R^2 \\ 0, \; if \; (x^2+y^2) > R^2 \end{cases}$$

for some value of c.

1. Determine c.
2. Find the marginal density function of X and Y.
3. Compute the probability that D, the distance from the origin of the point selected, is less than or equal to a.
4. Find E[D].

1.2 Problem

The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} e^{-(x+y)}, \; if \, 0 < x < \infty, 0 < y < \infty \\ 0, \; o/w \end{cases}$$

1.3 The Multinomial distribution

Suppose that a fair die is rolled 9 times. What is the probability that 1 appears three times, 2 and 3 twice each, 4 and 5 once each, and 6 not at all?

$$\frac{9!}{3!2!2!1!1!0!}\left(\frac{1}{6}\right)^3\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1$$

A sequence of n independent and identical experiments is performed. Suppose that each experiment can result in any one of r possible outcomes, with respective probabilities $p_1, p_2, \ldots, p_r, \sum_{i=1}^{r} p_i = 1$. If we let X_i denote the number of the n experiments that result in outcome number i, then

$$P(X_1 = n_1, X_2 = n_2, \ldots, X_r = n_r) = \frac{n!}{n_1! n_2! \ldots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

where $\sum_{i=1}^{r} n_i = n$. Here $(X_1, X_2, \ldots, X_r)$ is said to follow a multinomial distribution with parameters $(n, p_1, p_2, \ldots, p_r)$. For $r = 2$, we get the binomial distribution.

2 Independent Random Variables

Random variables X and Y are said to be independent if for two sets A and B of real numbers

$$P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$$

For discrete random variables

$$p(x, y) = p_X(x) p_Y(y)$$

and for continuous random variables

$$f(x, y) = f_X(x) f_Y(y)$$

2.1 Examples

1. Suppose that $n + m$ independent trials, having a common success probability p, are performed. If X is the number of successes in the first n trials, and Y is the number of successes in the final m trials, then X and Y are independent, since knowing the number of successes in the first n trials does not affect the distribution of the number of successes in the final m trials (by the assumption of independent trials).