Jointly distributed random variables

November 22, 2012 Debdeep Pati

1 Sum of two independent random variables

1. $X \sim \text{Bin}(n,p)$, $Y \sim Bin(m,p)$, then $X+Y \sim Bin(m+n,p)$
2. $X \sim Poiss(\lambda), Y \sim Poiss(\mu), X+Y \sim Poiss(\lambda+\mu)$
3. Suppose that the number of people that enter a post office on a given day is a Poisson random variable with parameter λ. Show that if each person that enters the post office is a male with probability p and a female with probability 1-p, then the number of males and females entering the post office are independent Poisson random variables with respective parameters λp and $\lambda(1-p)$.

   (Let X and Y denotes the number of males and females that enter the post office. Then

$$
\begin{aligned}
P(X=i,Y=j) &= P(X=i,Y=j \mid X+Y=i+j)P(X+Y=i+j) + \\
&\quad P(X=i,Y=j \mid X+Y \neq i+j)P(X+Y \neq i+j) \\
&= P(X=i,Y=j \mid X+Y=i+j)P(X+Y=i+j)
\end{aligned}
$$

   Since $(X+Y) \sim Poiss(\lambda), P(X+Y=i+j) = \exp\{-\lambda\}\lambda^{i+j}/(i+j)!$ and

$$
P(X=i,Y=j \mid X+Y=i+j) = \binom{i+j}{i} p^i (1-p)^j
$$

   Hence

$$
\begin{aligned}
P(X=i,Y=j) &= \binom{i+j}{i} p^i (1-p)^j \exp\{-\lambda\}\lambda^{i+j}/(i+j)! \\
&= e^{-\lambda p}\frac{(\lambda p)^i}{i!} e^{-\lambda(1-p)}\frac{(\lambda(1-p))^j}{j!}
\end{aligned}
$$

   Hence $X \sim Poisson(\lambda p)$ and $Y \sim Poisson(\lambda(1-p))$.)
4. $X_1 \sim N(\mu_1,\sigma_1^2), Y_1 \sim N(\mu_2,\sigma_2^2), X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.
5. $X_1 \sim N(\mu_1,\sigma_1^2), Y_1 \sim N(\mu_2,\sigma_2^2), X-Y \sim N(\mu_1-\mu_2, \sigma_1^2+\sigma_2^2)$.

6. A club basketball team will play 44-game season. 26 of these games are against class A teams and 18 are against class B teams. Suppose that the team will win each game against class A team with probability .4, and against class B team with probability .7. Assume also that the results of the different games are independent. Approximate the probability that

   a) the team wins 25 games or more.

   b) the team wins more games against class A teams than it against class B teams.

   Let X_A and X_B denote, respectively, the number of games the team wins against class A teams and against class B teams. X_A and X_B are independent binomial random variables, and

$$E(X_A) = 26(0.4) = 10.4, \quad Var(X_A) = 26(0.4)(0.6) = 6.24$$
$$E(X_B) = 18(0.7) = 12.6, \quad Var(X_B) = 18(0.7)(0.3) = 3.78$$

   By the normal approximation to binomial, $X_A + X_B$ will approximately have normal distribution $N(23, 10.02)$. Then

$$\begin{aligned} P(X_A + X_B) \geq 25) &= P(X_A + X_B) \geq 24.5) \\ &= P(\frac{X_A + X_B - 23}{\sqrt{10.02}} \geq \frac{24.5 - 23}{\sqrt{10.02}}) \\ &\approx P(Z \geq \frac{1.5}{\sqrt{10.02}}) \\ &\approx 1 - P(Z < 0.4739) \\ &\approx 0.3178 \end{aligned}$$

   $X_A - X_B$ is approximately $N(-2.2, 10.02)$. Then

$$\begin{aligned} P(X_A - X_B \geq 1) &= P(X_A - X_B \geq 0.5) \\ &= P(\frac{X_A + X_B + 2.2}{\sqrt{10.02}} \geq \frac{0.5 + 2.2}{\sqrt{10.02}}) \\ &\approx P(Z \geq \frac{2.7}{\sqrt{10.02}}) \\ &\approx 1 - P(Z \leq 0.8530) \\ &\approx 0.1968 \end{aligned}$$

2 Conditional distributions: discrete case

For any two events E and F, the conditional probability of E given F is defined as

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

FOr discrete random variables, the conditional probability mass function of X given that $Y = y$ is

$$\begin{aligned} p_{X|Y}(x \mid y) &= P(X = x \mid Y = y) \\ &= \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x, y)}{p(y)}. \end{aligned}$$

Conditional cumulative distribution function can be similarly defined by

$$F_{X|Y}(x \mid y) = P(X \leq x \mid Y = y) = \sum_{a \leq x} p_{X|Y}(a \mid y)$$

2.1 Examples done in class

1. Suppose that $p(x, y)$, the joint probability mass function of X and Y, is given by $p(0, 0) = .4, p(0, 1) = .2, p(1, 0) = .1, p(1, 1) = .3$. Calculate the conditional probability mass function of X, given that$Y = 1$.

2. If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X, given that $X + Y = n$.