Jointly distributed random variables -Class 23

November 14, 2013 Debdeep Pati

1 Jointly continuous random variable

If X and Y are continuous, $f(x,y)$ is called joint probability density function of X and Y if for any subset $C \subset \mathbb{R}^2$,

$$P((X,Y) \in C) = \int\int_{(x,y)\in C} f(x,y)dxdy$$

1. Double Integral: $P(X \in A, Y \in B) = \int_A \int_B f(x,y)dxdy$
2. Distribution function: $F(a,b) = P(X \in (-\infty,a], Y \in (-\infty,b])) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)dxdy$
3. By differentiation $\frac{\partial^2}{\partial a \partial b}F(a,b) = f(a,b)$
4. Marginal distribution: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy, f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$

1.1 Problem

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, \, 0 < x < \infty, 0 < y < \infty \\ 0, \, o/w \end{cases}$$

Find the marginal densities of X and Y and determine whether they are independent.

1.2 Problem

Consider a circle of radius R and suppose that a point within the circle is randomly chosen in such a manner that all regions within the circle of equal area are equally likely to contain the point. (In other words, the point is uniformly distributed within the circle.) If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen, it follows, since (X, Y) is equally likely to be near each point in the circle, that the joint density function of X and Y is given by

$$f(x,y) = \begin{cases} c, \, if \, (x^2+y^2) \le R^2 \\ 0, \, if \, (x^2+y^2) > R^2 \end{cases}$$

for some value of c.

1. Determine c.
2. Compute the probability that D, the distance from the origin of the point selected, is less than or equal to a.
3. Find the density of D.

2 Sum of two independent random variables

1. $X_1 \sim N(\mu_1, \sigma_1^2), Y_1 \sim N(\mu_2, \sigma_2^2), X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.
2. $X_1 \sim N(\mu_1, \sigma_1^2), Y_1 \sim N(\mu_2, \sigma_2^2), X-Y \sim N(\mu_1-\mu_2, \sigma_1^2+\sigma_2^2)$.
3. A club basketball team will play 44-game season. 26 of these games are against class A teams and 18 are against class B teams. Suppose that the team will win each game against class A team with probability .4, and against class B team with probability .7. Assume also that the results of the different games are independent. Approximate the probability that

   a) the team wins 25 games or more.

   b) the team wins more games against class A teams than it against class B teams.

   Let X_A and X_B denote, respectively, the number of games the team wins against class A teams and against class B teams. X_A and X_B are independent binomial random variables, and

$$
\begin{aligned}
E(X_A) &= 26(0.4) = 10.4, \quad Var(X_A) = 26(0.4)(0.6) = 6.24 \\
E(X_B) &= 18(0.7) = 12.6, \quad Var(X_B) = 18(0.7)(0.3) = 3.78
\end{aligned}
$$

   By the normal approximation to binomial, $X_A + X_B$ will approximately have normal distribution $N(23, 10.02)$. Then

$$
\begin{aligned}
P(X_A + X_B) \geq 25) &= P(X_A + X_B) \geq 24.5) \\
&= P(\frac{X_A + X_B - 23}{\sqrt{10.02}} \geq \frac{24.5 - 23}{\sqrt{10.02}}) \\
&\approx P(Z \geq \frac{1.5}{\sqrt{10.02}}) \\
&\approx 1 - P(Z < 0.4739) \\
&\approx 0.3178
\end{aligned}
$$

$X_A - X_B$ is approximately $N(-2.2, 10.02)$. Then

$$\begin{aligned} P(X_A - X_B \geq 1) &= P(X_A - X_B \geq 0.5) \\ &= P(\frac{X_A + X_B + 2.2}{\sqrt{10.02}} \geq \frac{0.5 + 2.2}{\sqrt{10.02}}) \\ &\approx P(Z \geq \frac{2.7}{\sqrt{10.02}}) \\ &\approx 1 - P(Z \leq 0.8530) \\ &\approx 0.1968 \end{aligned}$$