STA 4442/5440 Midterm 2 Practice 1

November 7, 2012

Name:

FSUID:

Please sign the following pledge and read all instructions carefully before starting the exam.

Pledge: I have neither given nor received any unauthorized aid in completing this exam, and I have conducted myself within the guidelines of the University Honor Code.

Signature: ______________________________

INSTRUCTIONS:

- This is a closed-book, closed-notes exam. You may **not** refer to your notes, the text, or any other books. You may use a calculator.
- Total time is 70 minutes (11:05 A.M to 12:15 P.M.)
- **Show all work**, clearly and in order, if you want to receive full credit. When you use your calculator, explain all relevant mathematics. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- **Answer all the questions in the space provided. You may attach additional sheets if necessary.**
- This test has 6 problems and is worth 80 points. It is your responsibility to make sure that you have all of the problems.
- **Good luck!**

Prob. No.	Max Points	Earned Pts.
1	20	
2	10	
3	20	
4	10	
5	10	
6	10	

TOTAL: __________

Question 1. (20 pts.) In actuarial science, one of the models used for describing mortality is

$$f(x) = \begin{cases} Cx^2(100-x)^2, 0 \le x \le 100 \\ 0, \text{ otherwise} \end{cases}$$

where x denotes the age at which a person dies.

(a) Find the value of C.

b) Let A be the event "Person lives past 60." Find $P(A)$.

c) Find the expected mortality.

a) $1 = \int_{-\infty}^{\infty} f(x)\,dx = C \int_0^{100} x^2(100-x)^2 dx \quad , \quad \frac{x}{100} = t$

$= C \int_0^1 t^2 10^4 (1-t)^2 10^4 \, dt \, 10^2 = C \cdot 10^{10} \int_0^1 t^2 (1-t)^2 dt$

$= C\, 10^{10} \left[\int_0^1 (t^2 - 2t^3 + t^4)\, dt \right] = C \cdot 10^{10} \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right]$

$= \frac{C\, 10^{10}}{30}$

$C = \frac{30}{10^{10}} = \frac{3}{10^9}$

b) $P(X \geq 60) = C \int_{60}^{100} x^2 (100-x)^2 dx = C\, 10^{10} \int_{0.6}^{1} t^2 (1-t)^2 dt.$

$= C \cdot 10^{10} \left[\frac{1}{3}(1 - 0.36) - \frac{1}{2}(1-(0.6)^4) + \frac{1}{5}(1-(0.6)^5) \right]$

c) $E(X) = C \int_0^{100} x \cdot x^2 (100-x)^2 dx = C\, 10^{12} \int_0^1 t^3 (1-t)^2 dt$

$= C\, 10^{12} \left[\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right] = \frac{C \cdot 10^{12}}{60} = \frac{3}{10^9} \frac{10^{12}}{60} = 50$

Question 2. (10 pts.) X and Y are two discrete random variables taking values $-1, 0$ and $+1$ each with joint probability given by

Table 1: Joint probability Table

$Y \downarrow X \rightarrow$	-1	0	+1	Total
-1	0	1/4	0	1/4
0	1/4	0	1/4	1/2
+1	0	1/4	0	1/4
Total	1/4	1/2	1/4	1

a) Find marginal p.m.f of X and Y.

b) Find whether X and Y are independent or not.

a) X

-1	0	+1
1/4	1/2	1/4

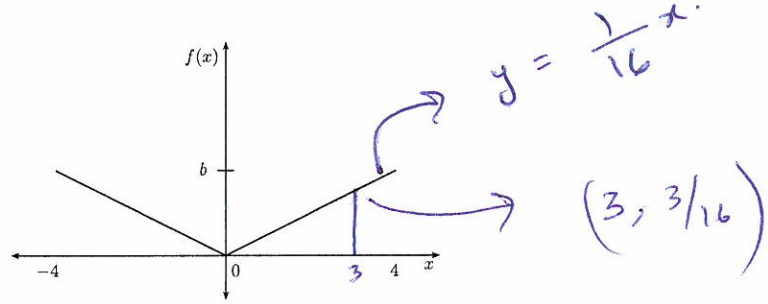
Y

-1	0	+1
1/4	$\frac{1}{2}$	1/4

b) $P(X=-1, Y=-1) = 0 \neq$

$P(X=-1) \cdot P(Y=-1) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

Question 3. (20 pts.) The figure is the probability density curve of the random variable X.



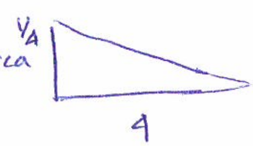
a) Find b so that $f(x)$ is a probability density function.

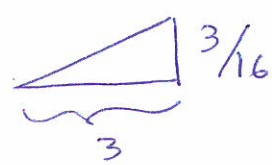
b) What is $P(-4 \le X \le 3)$?

c) What is $P(X = 1)$?

d) What is E(X)?

a) Area. $\left(\frac{1}{2} \times 4 \times b\right) \times 2 = 1 \Rightarrow b = 1/4$

b) $P(-4 \le X \le 3)$ = area 

+ area 

$$= \frac{1}{2} \times 4 \times \frac{1}{4} + \frac{1}{2} \times 3 \times \frac{3}{16}$$

$$= \frac{1}{2} + \frac{9}{32}$$

$$= \frac{25}{32}$$

c) $P(X=1) = 0$
(continuous density)

d) $E(X) = 0$ (Symmetric)

Question 4. (10 pts.) Two species are competing in a region for control of a limited amount of a certain resource. Let X =proportion of resource controlled by one species and suppose $X \sim \text{Unif}([0,1])$. Let $h(X) = \max(X, 1-X)$, then $h(X)$ is the amount of resource controlled by the superior species.

a) Find $E(h(X))$.

b) Find $Var(h(X))$.

$$h(x) = \max(x, 1-x)$$

$$= \begin{cases} 1-x & \text{if } 0 < x < 1/2 \\ x & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

$X \sim U(0,1)$

a) $$E(h(x)) = \int_0^{1/2} (1-x) \cdot 1 \cdot dx + \int_{1/2}^{1} x\, dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^{1/2} + \frac{x^2}{2}\Big|_{1/2}^{1}$$

$$= \frac{1}{2} - \frac{1}{8} + \frac{1}{2}\left[1 - 1/4\right] = 1 - \frac{1}{4} = 3/4$$

b) $$Var(h(x)) = E[h^2(x)] - \{E[h(x)]\}^2$$

$$= \int_0^{1/2} (1-x)^2 dx + \int_{1/2}^{1} x^2 dx - (3/4)^2$$

$$= \frac{7}{24} + \frac{7}{24} - \frac{9}{16} = \frac{1}{48}$$

Question 5. (10 pts.) Buses arrive at a specified stop at 15-minute intervals starting at 7 a.m. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that she waits

(a) less than 5 minutes for a bus.

(b) more than ten minutes for a bus.

a) P(less than 5 mins for a bus)

= P(arrives between 7:10 − 7:15)

+ P(arrives between 7:25 - 7:30)

$= \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$

b) P(more than 10 - mins for a bus)

= P(arrives between 7:00 - 7:05)

+ P(arrives between 7:15 - 7:20)

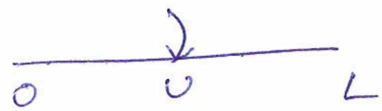
$= \frac{5+5}{30} = 1/3$

Question 6. (10 pts.) A point is picked randomly from the interval $[0, L]$.

(a) Define suitably a random variable X denoting the ratio of the length of the shorter and the longer interval formed.

(b) Find $P(X > 0.5)$.

(c) Find $E(X)$.



$U \sim U([0, L])$

a) $$X = \begin{cases} \dfrac{U}{L-U} & \text{if } U < \dfrac{L}{2} \\ \dfrac{L-U}{U} & \text{if } U > L/2 \end{cases}$$

b) $$P(X > 0.5) = P((X > 0.5) \cap (U < 0.5L)) + P((X > 0.5) \cap (U \geq L/2))$$

$$= P\left(\left(\frac{U}{L-U} > 0.5\right) \cap (U < L/2)\right) + P\left(\left(\frac{L-U}{U} > 0.5\right) \cap (U \geq L/2)\right)$$

$$= P\left(\frac{L}{3} < U < L/2\right) + P\left(\left(U \leq \frac{2L}{3}\right) \cap (U \geq L/2)\right)$$

$$= \int_{L/3}^{L/2} \frac{1}{L}\, du + \int_{L/2}^{2L/3} \frac{1}{L}\, du = \frac{1}{6} + 1/6 = 1/3$$

c) $$E(X) = \frac{1}{L}\left[\int_0^{L/2} \frac{x}{L-x}\, dx + \int_{L/2}^{L} \frac{L-x}{x}\, dx\right]$$

$$= \frac{1}{L}\left[\int_{L/2}^{L} \frac{L-z}{z}\, dz + \int_{L/2}^{L} \frac{L-x}{x}\, dx\right]$$

[by substituting $L-x = z$ in the first integral]

$$= \frac{2}{L}\left[L \log 2 - L/2\right] = 2(\log 2 - 0.5)$$