(Easier) Solution to 7.42 (a) (This is due to Shao Tang)

We have to minimise $Var\left(\sum_{i=1}^{k} a_i W_i\right)$ subject to $\sum_{i=1}^{k} a_i = 1$

Use Lagrangian. We have to minimise

$$L = Var\left(\sum_{i=1}^{k} a_i W_i\right) + \lambda\left(\sum_{i=1}^{k} a_i - 1\right) \quad \text{w.r.t} \quad \lambda, a_1, \ldots, a_n$$

$$= \sum_{i=1}^{k} a_i^2 \sigma_i^2 + \lambda\left(\sum_{i=1}^{k} a_i - 1\right)$$

$$\frac{\partial L}{\partial \lambda} = 0$$

$$\Rightarrow \sum_{i=1}^{k} a_i = 1$$

$$\frac{\partial L}{\partial a_i} = 0 \Rightarrow 2 \cancel{\Sigma}\, a_i \sigma_i^2 + \lambda \cancel{a_i} = 0$$

$$\Rightarrow a_i = \frac{-\lambda}{2\sigma_i^2}, \quad i = 1, 2, \ldots, k$$

$$\sum_{i=1}^{k} a_i = 1 \Rightarrow -\lambda \sum_{i=1}^{k} \frac{1}{2\sigma_i^2} = 1$$

$$\Rightarrow \lambda = -\frac{1}{\sum_{i=1}^{k} \frac{1}{2\sigma_i^2}}$$

$$\Rightarrow a_i = -\frac{\lambda}{2\sigma_i^2} = \frac{\frac{1}{\sigma_i^2}}{\sum_{i=1}^{k} \frac{1}{\sigma_i^2}} \quad \text{(Proved)}$$

Proof of $(I)^{-1}_{ii} \geq (I_{ii})^{-1}$ where I is Fisher Information

I is a symmetric positive definite matrix.

So I has the following spectral decomposition

$$I_{K\times K} = U_{K\times K} D_{K\times K} U^T_{K\times K}$$

where D is a diagonal matrix with eigen values $0 < d_1, d_2, \ldots, d_K$

U is an orthonormal matrix with the eigen vectors corresponding to $d_1, \ldots, d_K$. $U^T U = U U^T = I$

$$I^{-1} = (U^T)^{-1} D^{-1} U^{-1} = U D^{-1} U^T$$

Let $U = [U_1 : U_2 : \cdots : U_K]$ where U_j is the jth column of U

We can write

$$I = \sum_{j=1}^{K} d_j U_j U_j^T \quad \& \quad I^{-1} = \sum_{j=1}^{K} d_j^{-1} U_j U_j^T$$

Recall Cauchy-Schwarz Inequality

$$\left(\sum_{i=1}^{K} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{K} x_i^2\right)\left(\sum_{i=1}^{K} y_i^2\right) \longrightarrow (*)$$

~~Applying~~ Note $(I^{-1})_{ii} = \sum_{j=1}^{K} d_j^{-1} U_{ji}^2$ (U_{ji} is the ith element of U_j)

$$\& \quad (I)_{ii} = \sum_{j=1}^{K} d_j U_{ji}^2$$

Apply $(*)$ with $x_i = \sqrt{d_j^{-1}}\, U_{ji}$ & $y_i = \sqrt{d_j}\, U_{ji}$

$$\Rightarrow \sum_{j=1}^{K} d_j^{-1} U_{ji}^2 \sum_{j=1}^{K} d_j U_{ji}^2 \geq \sum_{j=1}^{K} U_{ji}^2$$

Since $U^T U = I$

$$\|U_j\|^2 = 1$$

$$\Rightarrow \sum_{i=1}^{k} U_{ji}^2 = 1$$

$$\Rightarrow \sum_{j=1}^{k} d_j^{-1} U_{ji}^2 \geq \frac{1}{\sum_{j=1}^{k} d_j U_{ji}^2}$$

$$\Leftarrow\!\!\Rightarrow \quad (I^{-1})_{ii} \geq (I_{ii})^{-1}$$

$X_1, \ldots, X_n \sim N(0, \sigma^2)$. $H_0: \sigma^2 \leq \sigma_0^2 \quad (\Theta_0)$

$H_1: \sigma^2 > \sigma_0^2 \quad (\Theta_0^c)$

$$S = \sum_{i=1}^{n} \frac{X_i^2}{\sigma_0^2}$$

Show that $\sup_{\sigma^2 \in \Theta_0} P_{\sigma^2}(S \geq c') = P_{\sigma_0^2}(S \geq c')$

Note that $S = \sum_{i=1}^{n} \frac{X_i^2}{\sigma_0^2}$ (under P_{σ^2})

follows $\frac{\sigma^2 \chi_n^2}{\sigma_0^2}$ distribution.

For any $\sigma \leq \sigma_0$

$$P\left(\chi_n^2 \geq \frac{c' \sigma_0^2}{\sigma^2}\right) \leq P\left(\chi_n^2 \geq \frac{c' \sigma_0^2}{\sigma_0^2}\right)$$

$$\Leftrightarrow \quad P_{\sigma^2}(S \geq c') \leq P_{\sigma_0^2}(S \geq c')$$

Hence $\sup_{\sigma^2 \in \Theta_0} P_{\sigma^2}(S \geq c') = P_{\sigma_0}(S \geq c')$

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $\quad \theta = (\mu, \sigma^2)$ unknown.

Consider $\tau(\theta) = \mu^2$

Let $W \equiv \bar{X}^2 - \dfrac{S^2}{n}$ where $S^2 = \dfrac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

& $W_+ = \max(W, 0)$

Show that $E(W - \mu^2)^2 - E(W_+ - \mu^2)^2 > 0$

Note that

$$E\left[(W - \mu^2)^2 - (W_+ - \mu^2)^2\right]$$

$$= E\left[\{(W - \mu^2)^2 - (W_+ - \mu^2)^2\} I(W < 0)\right]$$

$$+ E\left[\underbrace{\{(W - \mu^2)^2 - (W_+ - \mu^2)^2\} I(W > 0)}_{0 \text{ since } W^+ = W \text{ on } \{W > 0\}}\right]$$

$$= E\left[\{(W - \mu^2)^2 - (0 - \mu^2)^2\} I(W < 0)\right]$$

$$= E\left[(W - \mu^2)^2 I(W < 0)\right] - \mu^4 \longrightarrow (*)$$

Now $(W - \mu^2)^2 > (\mu^2)^2$ if $W < 0$

So $E\{(W - \mu^2)^2 I(W < 0)\} > \mu^4$

Hence $(*) > 0$

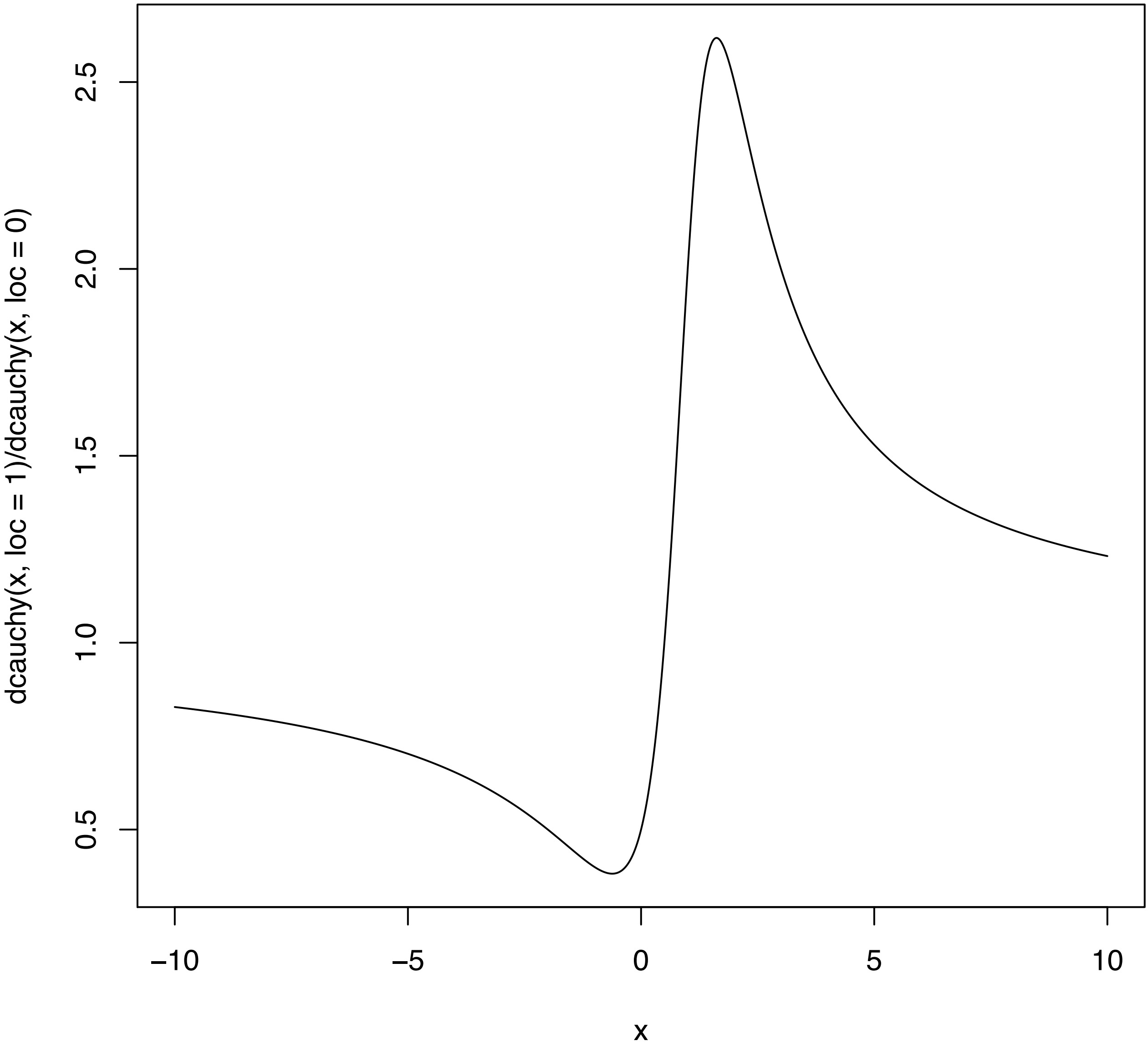
$H_0: \theta = \theta_0 \quad H_1: \theta = \theta_1 \; (> \theta_0)$

If $X \sim$ Cauchy (θ), the most powerful tests have rejection region

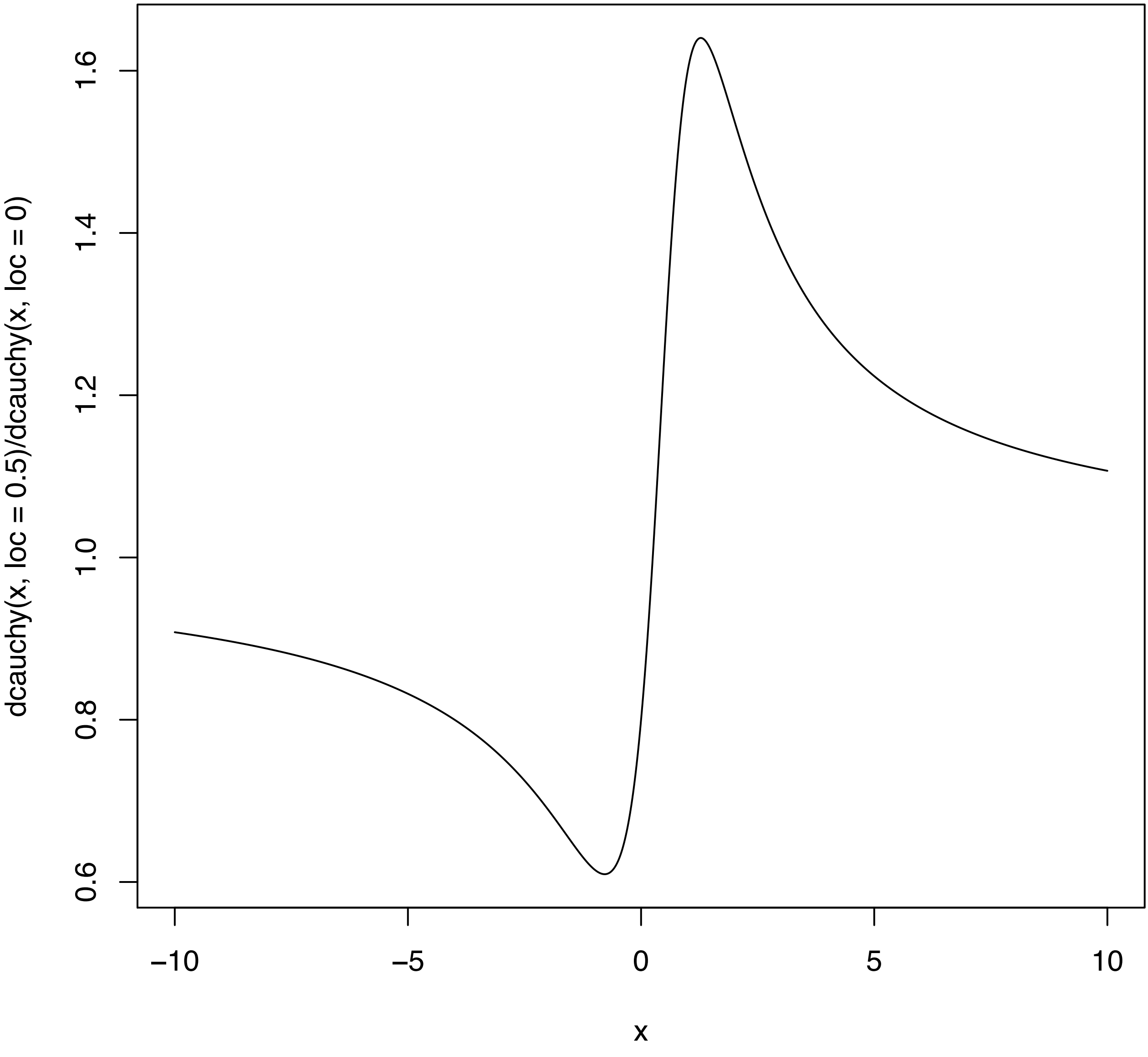
$$R = \{x : LR(x) > k\} = \{x : a^* < x < b^*\}$$

Here a^* & b^* depends on both θ_0 & θ_1

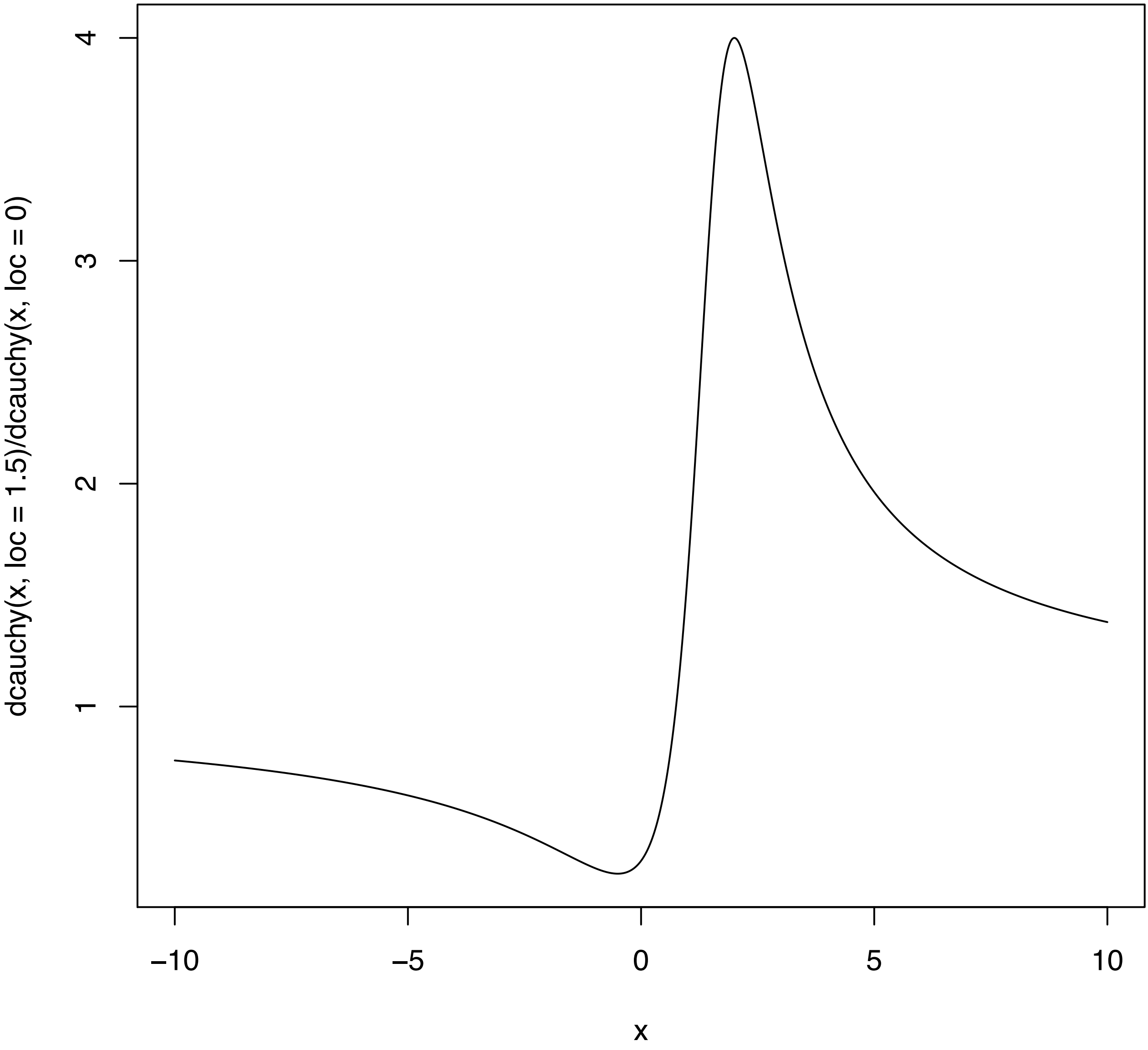
See the plots of LR (in R) made by Dr. Huffer for $\theta_0 = 0$ and θ_1 taking different values.

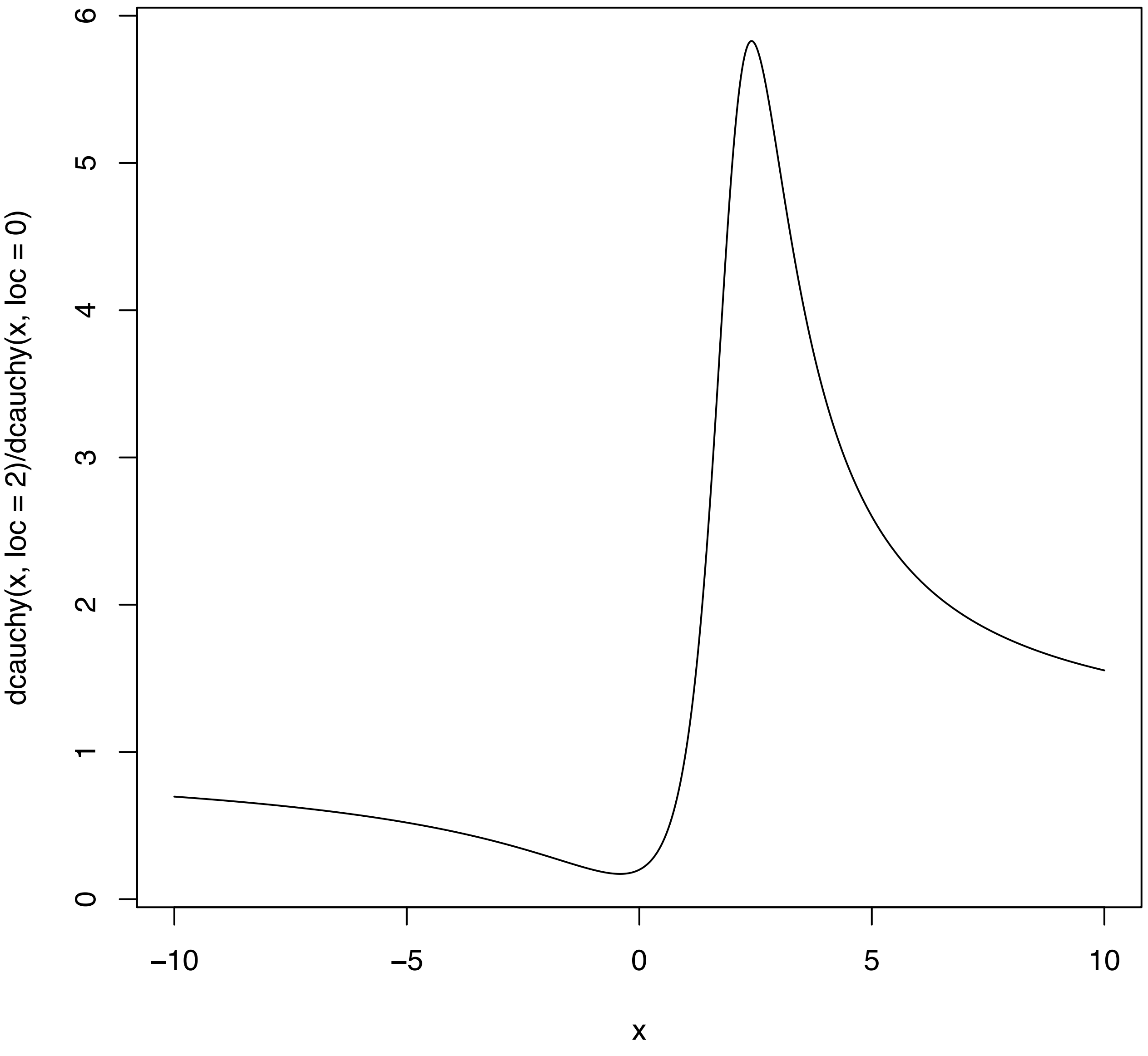


dcauchy(x, loc = 1)/dcauchy(x, loc = 0)
0.5
1.0
1.5
2.0
2.5
−10
−5
0
5
10
x



1.6
1.4
1.2
1.0
0.8
0.6
−10
−5
0
5
10
x
dcauchy(x, loc = 0.5)/dcauchy(x, loc = 0)


dcauchy(x, loc = 1.5)/dcauchy(x, loc = 0)
1
2
3
4
−10
−5
0
5
10
x



dcauchy(x, loc = 2)/dcauchy(x, loc = 0)
0
1
2
3
4
5
6
−10
−5
0
5
10
x

X_i's are independent & identically distributed continuous r.v's with mean μ and variance σ^2. Assume X_i's are symmetric about μ. Then $E(\text{Median}) = \mu$ where Median = median of $X_1, \ldots, X_n$.

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$

Case 1: n = odd $(=2m+1)$ Median $= M = X_{(m+1)}$

Density of $M = f_M(x) = (m+1)\binom{2m+1}{m} f_X(x)\, F_X(x)^m\, (1-F_X(x))^m$

where f_X is the common density of X_i's & F_X is the cdf.

Since X_i's are symmetric, $F_X(x) = 1 - F_X(2\mu - x)$ and $f_X(x) = f_X(2\mu - x)$, we have $f_M(x) = f_M(2\mu - x)$

and Hence $E(M) = E(2\mu - M) \Rightarrow E(M) = \mu$

Case 2: $n = 2m$

$$M = \frac{1}{2}\left(X_{(m)} + X_{(m+1)}\right)$$

The joint prob. density for $f_{X_{(m)}, X_{(m+1)}}(x, y) =$

$$m^2 \binom{2m}{m} f_X(x_1)\, f_X(x_2)\, F_X(x_1)^{m-1} (1-F_X(x_2))^{m-1} \quad x_1 \leq x_2$$

Since $f_{X_{(m)}, X_{(m+1)}}(x, y) = f_{X_{(m)}, X_{(m+1)}}(2\mu - y, 2\mu - x)$

$$E(M) = E\left(\frac{X_{(m)} + X_{(m+1)}}{2}\right) = E\left(\frac{2\mu - X_{(m+1)} + 2\mu - X_{(m)}}{2}\right)$$

$$= E(2\mu - M)$$

$\Rightarrow E(M) = \mu$.

	$x=0$	1	2	3	4	5
$f(x\|0)$	0.05	0.1	0.15	0	0.5	0.2
$f(x\|1)$	0.15	0.4	0.3	0.05	0.05	0.05

Show that $R=\{1\}$ does not ~~verify~~ satisfy assumptions of NP Lemma

$X=3$	1	0	2	5	4
$LR(\underset{\sim}{x}) = \infty$	4	3	2	$\frac{1}{4}$	$\frac{1}{10}$

We cannot find a k s.t

$$x \in R \iff \{LR(x) > k\}$$

Whatever k you choose,

$\{LR(x) > k\}$ will be of the form

$\{4, 5, 2, 0, 1, 3\}$ OR

$\{5, 2, 0, 1, 3\}$ OR

$\{2, 0, 1, 3\}$ OR

$\{0, 1, 3\}$ OR

$\{1, 3\}$

$\{3\}$

So $R = \{1\}$ does not satisfy NP-lemma

$T \sim \text{Poisson}(n\lambda)$

$$E\left(\frac{n}{T+1}\right) = \sum_{k=0}^{\infty} \frac{n}{k+1} \frac{e^{-n\lambda}(n\lambda)^k}{k!}$$

$$= \frac{1}{\lambda}\left(1-e^{-n\lambda}\right)$$

$$\sum_{k=0}^{\infty} \frac{n}{k+1} e^{-n\lambda} \frac{(n\lambda)^k}{k!}$$

$$= n e^{-n\lambda} \sum_{k=0}^{\infty} \frac{(n\lambda)^k}{(k+1)!}$$

$k+1=j$

$$= n e^{-n\lambda} \sum_{j=1}^{\infty} \frac{(n\lambda)^{j-1}}{j!}$$

$$= \frac{n e^{-n\lambda}}{n\lambda} \sum_{j=1}^{\infty} \frac{(n\lambda)^j}{j!}$$

$$= \frac{\cancel{n} e^{-n\lambda}}{\cancel{n}\lambda} \left[e^{n\lambda} - 1 \right]$$

$$= \frac{1}{\lambda}\left(1-e^{-n\lambda}\right).$$