ANOVA continued

Table 12.1 FEF data for smoking and nonsmoking males

Group number, i	Group name	Mean FEF (L/s)	*sd* FEF (L/s)	n_i
1	NS	3.78	0.79	200
2	PS	3.30	0.77	200
3	NI	3.32	0.86	50
4	LS	3.23	0.78	200
5	MS	2.73	0.81	200
6	HS	2.59	0.82	200

Source: Reprinted by permission of the *New England Journal of Medicine, 302*(13), 720–723, 1980.

1. Test if the mean FEF scores differ significantly among the six groups in Table 12.1.

2. We can compute the between SS as 184.38 and within SS = 663.87. There are 1050 observations combined over all 6 groups. Between MS = 184.38/5 = 36.875. Within MS = 663.87/(1050-6) = 663.87/1044 = 0.636. F = Between MS/Within MS = 36.875/0.636 = 58.0 $\sim F_{5,1044}$ under null hypothesis. $F_{5,1044,.999} = qf(.999, 5, 1044) = 4.13 < 58.0 = F$. $p < 0.001$. At least two of the means are significantly different.

Comparison of Specific Groups in One-Way ANOVA

The previous test does not tell us which of the groups have means that differ from each other. The usual practice is to perform the F test first and if null hypothesis is rejected, then specific groups are compared. We want to test if groups 1 and 2 have means that are significantly different from each other.

$\bar{Y}_1$ is normally distributed with mean $\mu + \alpha_1$ and variance σ^2/n_1 and $\bar{Y}_2$ is normally distributed with mean $\mu + \alpha_2$ and variance σ^2/n_2.

$$\begin{aligned} \bar{Y}_1 - \bar{Y}_2 &\sim N(\alpha_1 - \alpha_2, \sigma^2(1/n_1 + 1/n_2)) \\ z &= \bar{Y}_1 - \bar{Y}_2 / \sqrt{\sigma^2(1/n_1 + 1/n_2)} \end{aligned}$$

Pooled Estimate of the Variance for One-Way ANOVA

$$s^2 = \sum_{i=1}^{k}(n_i - 1)s_i^2 \Big/ \sum_{i=1}^{k}(n_i - 1) = \left[\sum_{i=1}^{k}(n_i - 1)s_i^2\right] \Big/ (n-k) = \text{Within MS}$$

t **Test for the Comparison of Pairs of Groups in One-Way ANOVA (LSD Procedure)**

Suppose we wish to compare two specific groups, arbitrarily labeled as group 1 and group 2, among k groups. To test the hypothesis H_0: $\alpha_1 = \alpha_2$ versus H_1: $\alpha_1 \neq \alpha_2$, use the following procedure:

(1) Compute the pooled estimate of the variance s^2 = Within MS from the one-way ANOVA.

(2) Compute the test statistic

$$t = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

which follows a t_{n-k} distribution under H_0.

(3) For a two-sided level α test,

if $\quad t > t_{n-k,1-\alpha/2} \quad$ or $\quad t < t_{n-k,\alpha/2}$

then reject H_0

if $\quad t_{n-k,\alpha/2} \le t \le t_{n-k,1-\alpha/2}$

then accept H_0

Linear contrast

1. Compare the pulmonary function of the group of smokers who inhale cigarettes with the group of nonsmokers. The three groups of inhaling smokers could be combined to form one group of 600 inhaling smokers. However, these three groups were selected so as to be of the same size, whereas in the general population the proportions of light, moderate, and heavy smokers are not likely to be the same. Suppose large population surveys report that 70% of inhaling smokers are moderate smokers, 20% are heavy smokers, and 10% are light smokers. How can inhaling smokers as a group be compared with nonsmokers? A linear contrast (L) is any linear combination of the individual group means such that the linear coefficients add up to 0.

   $$L = \sum_{i=1}^{k} c_i \bar{y}_i, \text{ where } \sum_{i=1}^{k} c_i = 0.$$

   Linear contrast for the above example is $\bar{y}_1 - 0.1\bar{y}_4 - 0.7\bar{y}_5 - 0.2\bar{y}_6$.

2. Test the hypothesis that the underlying mean of the linear contrast defined in previous example is significantly different from 0. $s^2 = 0.636$. $L = \bar{y}_1 - 0.1\bar{y}_4 - 0.7\bar{y}_5 - 0.2\bar{y}_6 = 1.03$. $se(L) = \sqrt{s^2 \sum_{i=1}^{k} c_i^2/n_i} = 0.070.t = L/se(L) = 1.03/0.070 = 14.69 \sim t_{1044}.qt(0.999, 1044) = 3.10$. $p < 0.001$. The linear contrast is very highly significant and the inhaling smokers as a group have much worse pulmonary function than the nonsmokers.

3. Linear contrast can also be used when the different groups correspond to different dose levels of a particular quantity, and the coefficients of the contrast are chosen to reflect a particular dose-response relationship. Suppose we want to study whether or not the number of cigarettes smoked is related to the level of FEF among those smokers who inhale cigarette smoke. We can focus on the light smokers, moderate smokers, and heavy smokers. Light smokers smoke from 1 to 10 cigarettes per day (5.5 on average) Moderate smokers smoke from 11 to 39 per day (25 on average) Heavy smokers smoke at least 40 a day (assume they smoke exactly 40)

$$L = \sum_{i=1}^{k} c_i \overline{y}_i \quad \text{where } \sum_{i=1}^{k} c_i = 0$$

Because $Var(\overline{y}_i) = s^2/n_i$,

$$Var(L) = s^2 \sum_{i=1}^{k} c_i^2/n_i$$

***t* Test for Linear Contrasts in One-Way ANOVA** Suppose we want to test the hypothesis H_0: $\mu_L = 0$ versus H_1: $\mu_L \neq 0$, using a two-sided test with significance level $= \alpha$, where $y_{ij} \sim N(\mu + \alpha_i, \sigma^2)$, $\mu_L = \sum_{i=1}^{k} c_i \alpha_i$, and $\sum_{i=1}^{k} c_i = 0$.

(1) Compute the pooled estimate of the variance = s^2 = Within MS from the one-way ANOVA.

(2) Compute the linear contrast

$$L = \sum_{i=1}^{k} c_i \overline{y}_i$$

(3) Compute the test statistic

$$t = \frac{L}{\sqrt{s^2 \sum_{i=1}^{k} \frac{c_i^2}{n_i}}}$$

(4) If $t > t_{n-k,1-\alpha/2}$ or $t < t_{n-k,\alpha/2}$ then reject H_0.

If $t_{n-k,\alpha/2} \le t \le t_{n-k,1-\alpha/2}$ then accept H_0.

(5) The exact *p*-value is given by

$$L = 5.5\overline{y}_4 + 25\overline{y}_5 + 40\overline{y}_6 \Rightarrow L = (5.5 - 23.5)\overline{y}_4 + (25 - 23.5)\overline{y}_5 + (40 - 23.5)\overline{y}_6$$
$$= -18\overline{y}_4 + 1.5\overline{y}_5 + 16.5\overline{y}_6 = -11.31$$

$$se(L) = \sqrt{0.636\left[\frac{(-18)^2}{200} + \frac{1.5^2}{200} + \frac{16.5^2}{200}\right]} = 1.38$$

$$t = L / se(L) = -11.31/1.38 = -8.20 \sim t_{1044} \text{ under } H_0$$

$$qt(.99,1044) = 2.33$$

The trend is very highly significant. Among smokers who inhale, the greater the number of cigarettes smoked per day, the worse the pulmonary function.

Multiple Comparisons

When there are a large number of groups and every pair of groups is compared, then some significant differences are likely to be found just by chance. Suppose there are 10 groups. There are $\binom{10}{2} = 45$ possible pairs of groups to be compared. Using a 5 % level of significance would imply that $.05(45) = 2.25$, or about two comparisons, are likely to be significant by chance alone. Multiple-comparisons procedures ensure that the overall probability of declaring any significant differences between all possible pairs of groups is maintained at some fixed significance level.

Comparison of Pairs of Groups in One-Way ANOVA—Bonferroni Multiple-Comparisons Procedure Suppose we wish to compare two specific groups, arbitrarily labeled as group 1 and group 2, among k groups. To test the hypothesis H_0: $\alpha_1 = \alpha_2$ versus H_1: $\alpha_1 \neq \alpha_2$, use the following procedure:

(1) Compute the pooled estimate of the variance s^2 = Within MS from the one-way ANOVA.

(2) Compute the test statistic

$$t = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

(3) For a two-sided level α test, let $\alpha^* = \alpha \Big/ \binom{k}{2}$.

If $t > t_{n-k,1-\alpha^*/2}$ or $t < t_{n-k,\alpha^*/2}$ then reject H_0

if $t_{n-k,\alpha^*/2} \le t \le t_{n-k,1-\alpha^*/2}$ then accept H_0

In a study of k groups, there are $\binom{k}{2}$ possible two-group comparisons.

α^* = significance level that each two-group comparison is conducted.

E: event that at least one of the two-group comparison is statistically significant.

$P(E)$: experiment-wise type I error.

Determine α^* such that $P(E) = \alpha$.

P(none of the two-group comparisons is statistically significant) $= P(\bar{E}) = 1-\alpha$.

If each of the two-group comparisons were independent, then $P(\bar{E}) = (1-\alpha^*)^c$,

where $c = \binom{k}{2}$.

$(1-\alpha) = (1-\alpha^*)^c$

When α^* is small, $(1-\alpha^*)^c \approx 1 - c\alpha^*$

$1-\alpha = 1 - c\alpha^*$

$\alpha^* = \alpha \Big/ \binom{k}{2}$.

Bonferroni procedure is conservative.