Properties of Expectation - Class 25

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1 Properties of expectation

1. If n balls are randomly selected from an urn containing N balls of which m are white, find the expected number of white balls selected. Let X denote the number of white balls selected, $X = X_1 + \cdots + X_m$ where

$$X_i = \begin{cases} 1, \text{ if the ith white ball is selected} \\ 0, \text{ otherwise} \end{cases}$$

$E(X_i) = P(X_i = 1) = \frac{\binom{1}{1}\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N}$ Hence $E(X) = mn/N$.

Alternatively, $X = Y_1 + \cdots + Y_n$ where

$$Y_i = \begin{cases} 1, \text{ if the ith ball selected is white} \\ 0, \text{ otherwise} \end{cases}$$

Since the i ball selected is equally likely to be any of the N balls

$$E(Y_i) = \frac{m}{N}$$

So $E(X) = mn/N$

2. A group of N people throw their hats into the center of a room. The hats are mixed up, and each person randomly selects one. Find the expected number of people that select their own hat. Let X denote the number of matches, we can $E(X)$ by $X = X_1 + X_2 + \ldots X_N$ where

$$X_i = \begin{cases} 1 \text{ if the ith person selects his own hat} \\ 0 \, \text{otherwise} \end{cases}$$

Since, for each i, the ith person is equally likely to select any of the N hats

$$E(X_i) = P(X_i = 1) = \frac{1}{N}$$

Thus

$$E(X) = E(X_1) + \ldots + E(X_N) = \frac{1}{N} \times N = 1$$

On an average, exactly one person selects his own hat.

3. Suppose that there are N different types of coupons and each time one obtains a coupon it is equally likely to be any one of the N types. Find the expected number of coupons one need amass before obtaining a complete set of at least one of each type.

   Let X denote the number of coupons collected before a complete set is attained. We define $X_i, i = 0, 1, \ldots. N-1$ to be the number of additional coupons that need to be obtained after i distinct types have ten collected in order to obtain another type, and note that $X = X_0 + X_1 + \ldots, X_{N-1}$. When i distinct types of coupons have already been collected, a new coupon obtained will be of a distinct type with probability $(N-i)/N$. Hence

$$P(X_i = k) = \frac{N-i}{N}\left(\frac{i}{N}\right)^{k-1}, k \geq 1$$

   Clearly, X_i is a geometric random variable with parameter $(N-i)/N$. Hence $E(X_i) = \frac{N}{N-i}$ Thus

$$E(X) = 1 + \frac{N}{N-1} + \frac{N}{N-2} + \ldots + N = N\left(1 + \ldots + \frac{1}{N-1} + \frac{1}{N}\right)$$

1.1 Moments of the number of Events that occur

1. Concept of moment is evolved from the moment concept in physics.

2. The nth moment (about zero) of a probability density function f(x) is the expected value of $E(X^n)$.

3. For discrete random variables with probability mass function $p(x)$. $E(X^n) = \sum_{allx} x^n p(x)$

4. For continuos random variables with probability density function $f(x)$. $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx$

5. $E[X]$ is the first moment of X, $E[X^2]$ is the second moment of the pdf of X and so on.

6. For given events $A_1, \ldots, A_n$, find $E[X]$, where X is the number of these events that occur. We define an indicator variable I_i for event A_i, and $I_i = 1$ if A_i occurs. As $X = \sum_{i=1}^{n} I_i$, we have $E(X) = E(\sum_{i=1}^{n} I_i) = \sum_{i=1}^{n} E(I_i) = \sum_{i=1}^{n} P(A_i)$.

7. We are interested in the number of pairs of events that occurs, the number of pairs

is equal to $\sum_{i<j} I_i I_j$, which is equal to $\binom{X}{2}$. Hence

$$\begin{aligned} E\left[\binom{X}{2}\right] &= \sum_{i<j} E(I_i I_j) = \sum_{i<j} P(A_i \cap A_j) \\ E(X^2 - X) &= 2\sum_{i<j} P(A_i \cap A_j) \end{aligned}$$

8.

$$\begin{aligned} \binom{X}{k} &= \sum_{i_1<i_2<\ldots i_k} I_{i_1} I_{i_2} \ldots I_{i_k} \\ E\left[\binom{X}{k}\right] &= \sum_{i_1<i_2<\cdots<i_k} E(I_{i_1} I_{i_2} \ldots I_{i_k}) = \sum_{i_1<i_2<\ldots<i_k} P(A_{i_1} A_{i_2} \ldots \cap A_{i_k}) \end{aligned}$$

1.2 Moments of Binomial random variable

Consider n independent trials, with each trial being a success with probability p. Let A_i be he event that the trial i is a success. When $i \neq j, P(A_i \cap A_j) = p^2$. Clearly $E((\binom{X}{2}) = \binom{n}{2}p^2$ and hence $E(X^2) = n(n-1p^2 + np$. $E(X(X-1)(X-2)) = n(n-1)(n-2)p^3$ and hence $E(X^3) = 3E(X^2) - 2E(X) + n(n-1)(n-2)p^3 = 3n(n-1)p^2 + np + n(n-1)(n-2)p^3$.