Properties of Expectation- Class 26

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1 Moments of the number of Events that occur

1. Concept of moment is evolved from the moment concept in physics.
2. The nth moment (about zero) of a probability density function f(x) is the expected value of $E(X^n)$.
3. For discrete random variables with probability mass function $p(x)$. $E(X^n) = \sum_{allx} x^n p(x)$
4. For continuos random variables with probability density function $f(x)$. $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx$
5. $E[X]$ is the first moment of X, $E[X^2]$ is the second moment of the pdf of X and so on.
6. For given events $A_1, \ldots, A_n$, find $E[X]$, where X is the number of these events that occur. We define an indicator variable I_i for event A_i, and $I_i = 1$ if A_i occurs. As $X = \sum_{i=1}^n I_i$, we have $E(X) = E(\sum_{i=1}^n I_i) = \sum_{i=1}^n E(I_i) = \sum_{i=1}^n P(A_i)$.
7. We are interested in the number of pairs of events that occurs, the number of pairs is equal to $\sum_{i<j} I_i I_j$, which is equal to $\binom{X}{2}$. Hence

$$\begin{aligned} E\left[\binom{X}{2}\right] &= \sum_{i<j} E(I_i I_j) = \sum_{i<j} P(A_i \cap A_j) \\ E(X^2 - X) &= 2\sum_{i<j} P(A_i \cap A_j) \end{aligned}$$

8.

$$\begin{aligned} \binom{X}{k} &= \sum_{i_1<i_2<\ldots i_k} I_{i_1} I_{i_2} \ldots I_{i_k} \\ E\left[\binom{X}{k}\right] &= \sum_{i_1<i_2<\cdots<i_k} E(I_{i_1} I_{i_2} \ldots I_{i_k}) = \sum_{i_1<i_2<\ldots<i_k} P(A_{i_1} A_{i_2} \ldots \cap A_{i_k}) \end{aligned}$$

1.1 Moments of Binomial random variable

Consider n independent trials, with each trial being a success with probability p. Let A_i be he event that the trial i is a success. When $i \neq j, P(A_i \cap A_j) = p^2$. Clearly $E(\binom{X}{2}) = \binom{n}{2}p^2$ and hence $E(X^2) = n(n-1p^2 + np$. $E(X(X-1)(X-2)) = n(n-1)(n-2)p^3$ and hence $E(X^3) = 3E(X^2) - 2E(X) + n(n-1)(n-2)p^3 = 3n(n-1)p^2 + np + n(n-1)(n-2)p^3$.

2 Covariance, Variance of Sums and Correlations

Theorem 1 *If X and Y are independent, then for any function h and g*

$$E(g(X)h(Y)) = E(g(X))E(h(Y))$$

The covariance between X and Y, denoted by $Cov(X,Y)$, is defined by

$$\begin{aligned} Cov(X,Y) &= E[(X-E[X])(Y-E[Y])] \\ &= E[XY - E[X]Y - XE[Y] + E[Y]E[X]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

If X and Y are independent, $Cov(X,Y) = 0$. If $Cov(X,Y) = 0$, X and Y may not be independent. $P(X=0) = P(X=1) = P(X=-1) = 1/3$ $Y = 0$ if $X \neq 0$, $Y = 1$ if $X = 0$ $XY = 0$, $E[XY] = 0$, and $E[X] = 0$, so $Cov(X,Y) = E[XY] - E[X]E[Y] = 0$ But X and Y are not independent.

Theorem 2

1. $Cov(X,Y) = Cov(Y,X)$
2. $Cov(X,X) = Var(X)$
3. $Cov(aX,Y) = aCov(X,Y)$
4. $Cov(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) = \sum_{i=1}^n \sum_{j=1}^m Cov(X_i, X_j)$

Hence

$$\begin{aligned} Var(\sum_{i=1}^n X_i) &= Cov(\sum_{i=1}^n X_j, \sum_{i=1}^n X_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n Cov(X_i, X_j) \\ &= \sum_{i=1}^n Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j) \end{aligned}$$

which is equivalent to the following

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

If $X_1, \ldots, X_n$ are pairwise independent, we have

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$$