Properties of Expectation - Class 27

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1 Correlation of two random variables

Correlation of two random variables X and Y , denoted by $\rho(X,Y)$, and is defined by

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Theorem 1 $-1 \leq \rho(X,Y) \leq 1$

$$\begin{aligned} 0 &\leq Var(X/\sigma_x + Y/\sigma_y) \\ &= 2(1+\rho(X,Y)) \end{aligned}$$

Showing that $\rho(X,Y) \geq -1$ and using

$$\begin{aligned} 0 &\leq Var(X/\sigma_x - Y/\sigma_y) \\ &= 2(1-\rho(X,Y)) \end{aligned}$$

showing that $\rho(X,Y) \leq 1$.

Example: Let I_A and I_B be indicator variables for the event A and B respectively. Then

$$\begin{aligned} Cov(I_A, I_B) &= P(A \cap B) - P(A)P(B) \\ &= P(A \mid B)P(B) - P(A)P(B) \\ P(B)[P(A \mid B - P(A)] & \end{aligned}$$

2 Conditional Expectation

Define the conditional probability function

$$p_{X|Y}(x \mid y) = \frac{p(x,y)}{p_Y(y)}$$

1. Conditional expectation in the discrete case: $E(X \mid Y = y) = \sum_x x p_{X|Y}(x \mid y)$

2. If X and Y are joint continuous, with a joint probability density function $f(x, y)$, the conditional expectation is defined by

$$E(X \mid Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) dx$$

Find $E(X \mid Y)$.

3. Denote $E[X \mid Y]$ the function of random variable Y whose value at $Y = y$ is $E[X \mid Y = y]$. $E[X|Y = y]$ is itself a random variable.

4. Result: $E(X) = E(E(X \mid Y))$.

5. A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?
Let X denote the amount of time (in hours) until the miner reaches safety, and Y denote the door he initially choose. Then $P(Y = 1) = P(Y = 2) = P(Y = 3) = 1/3$. Observe that

$$\begin{aligned} E(X) = E(E(X \mid Y)) &= E(X \mid Y = 1)P(Y = 1) + E(X \mid Y = 2)P(Y = 2) + E(X \mid Y = 3)P(Y = 3) \\ &= \frac{1}{3}\{E(X \mid Y = 1) + E(X \mid Y = 2) + E(X \mid Y = 1)\} \\ &= \frac{1}{3}\{3 + 5 + E(X) + 7 + E(X)\} \end{aligned}$$

Hence $3E(X) = 15 + 2E(X)$ whence $E(X) = 15$.