Poisson Regression

Model: $X_1, \ldots, X_n$ are independent with $X_i \sim \text{Poisson}(\lambda_i)$ where $\lambda_i = \exp(\beta_0 + \beta_1 z_i)$ and $z_1, \ldots, z_n$ are known constants.

(a) Let $\theta = (\beta_0, \beta_1)$. The joint pmf is

$$\begin{aligned} f(\boldsymbol{x} \mid \theta) &= \prod_{i=1}^{n} \frac{\lambda_i^{x_i} e^{-\lambda_i}}{x_i!} = e^{-\sum_i \lambda_i} \left(\prod_i x_i!\right)^{-1} \exp\left\{\sum_i x_i(\beta_0 + \beta_1 z_i)\right\} \\ &= e^{-\sum_i \exp(\beta_0 + \beta_1 z_i)} \left(\prod_i x_i!\right)^{-1} \exp\left\{\beta_0 \sum_i x_i + \beta_1 \sum_i z_i x_i\right\} \end{aligned}$$

which has the form of an n-variate 2pef with natural sufficient statistic $T(\boldsymbol{X}) = (\sum_i X_i\,,\, \sum_i z_i X_i)$. Note that the parameter space is $\Theta = \mathbb{R}^2$.

(b) The MLE will be the value of θ which solves $E_\theta T(\boldsymbol{X}) = T(\boldsymbol{x})$ which is just an abbreviation for the system of equations

$$\begin{aligned} E \sum_i X_i &= \sum_i x_i \\ E \sum_i z_i X_i &= \sum_i z_i x_i\,. \end{aligned}$$

Since $EX_i = \exp(\beta_0 + \beta_1 z_i)$, this becomes

$$\begin{aligned} \sum_i \exp(\beta_0 + \beta_1 z_i) &= \sum_i x_i \\ \sum_i z_i \exp(\beta_0 + \beta_1 z_i) &= \sum_i z_i x_i \end{aligned} \qquad (\ddagger)$$

(c) From part (a), the log-likelihood is

$$\ell(\theta) = \log f(\boldsymbol{x} \mid \theta) = -\sum_i \exp(\beta_0 + \beta_1 z_i) + \beta_0 \sum_i x_i + \beta_1 \sum_i z_i x_i \; + \text{ constant}\,.$$

The MLE will (typically) be a stationary point, that is, a solution of the system

$$\begin{aligned} \frac{\partial \ell}{\partial \beta_0} &= -\sum_i \exp(\beta_0 + \beta_1 z_i) + \sum_i x_i \;=\; 0 \\ \frac{\partial \ell}{\partial \beta_1} &= -\sum_i z_i \exp(\beta_0 + \beta_1 z_i) + \sum_i z_i x_i \;=\; 0\,, \end{aligned}$$

which is clearly equivalent to ($\ddagger$) above.

Optional Remarks (not included in test material): You were not asked to solve the system ($\ddagger$) to obtain the MLE, but I will give a few remarks on this. The system cannot be solved completely in closed form, but some progress can be made. If $\sum x_i > 0$, dividing the second equation by the first leads to a single equation in one unknown

$$\frac{\sum z_i e^{\beta_1 z_i}}{\sum e^{\beta_1 z_i}} = \frac{\sum z_i x_i}{\sum x_i}\,.$$

The left side of this equation approaches $\max z_i$ as $\beta_1 \to \infty$ and $\min z_i$ as $\beta_1 \to -\infty$. So a finite solution for β_1 exists so long as

$$\min z_i < \frac{\sum z_i x_i}{\sum x_i} < \max z_i\,.$$