Practice Problems on Sufficiency

February 8, 2016 STA 5327

1. Let n items be drawn in order without replacement from a shipment of N items of which $N\theta$ are bad, where $0 < \theta < 1$. Let $X_i = 1$ if the ith item drawn is bad, and is 0 otherwise. Show that $\sum_{i=1}^n X_i$ is sufficient for θ directly from definition and also by the factorization theorem.

2. Suppose $X_1, \ldots, X_n$ is a sample from a population with one of the following densities.

   (a) $f(x \mid \theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0$. This is Beta$(\theta, 1)$ density.

   (b) $f(x \mid \theta) = \theta a x^{a-1} \exp(-\theta x^a), x > 1, \theta > 0, a > 0$. This is known as the Weibull Density.

   (c) $f(x \mid \theta) = \theta a^\theta / x^{(\theta+1)}, x > a, \theta > 0, a > 0$. This is known as the Pareto density.

   In all cases, find the real-valued sufficient statistic for θ and a fixed.

3. Let $X_1, X_2, \ldots, X_n$ be a sample from a population with density

$$f(x \mid \theta) = \begin{cases} a(\theta)h(x), & \text{if} \quad \theta_1 \le x \le \theta_2 \\ 0, & \text{otherwise} \end{cases}$$

   where $h(x) \ge 0$, $\theta = (\theta_1, \theta_2)$ with $-\infty < \theta_1 \le \theta_2 < \infty$ and

$$a(\theta) = \left(\int_{\theta_1}^{\theta_2} h(x)dx \right)^{-1}.$$

   Find a sufficient statistic for θ.

4. Let $\theta = (\theta_1, \theta_2)$ be a bivariate parameter. Suppose that $T_1(X)$ is sufficient for θ_1 whenever θ_2 is fixed and known, whereas $T_2(X)$ is sufficient for θ_2 whenever θ_1 is fixed and known. Assume that θ_1, θ_2 vary independently, $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$ and that the set $S = \{x : f(x \mid \theta) > 0\}$ does not depend on θ.

   (a) Show that if T_1 and T_2 do not depend on θ_2 and θ_1 respectively, then $(T_1(X), T_2(X))$ is sufficient for θ.

   (b) Exhibit an example in which $(T_1(X), T_2(X))$ is sufficient for θ, $T(X)$ is sufficient for θ_1 whenever θ_2 is fixed and known, but $T_2(X)$ is not sufficient for θ_2 , when θ_1 is fixed and known.

5. Let $X_1, X_2, \ldots, X_m; Y_1, Y_2, \ldots, , Y_n$, be independently distributed according to N(μ, σ^2) and N(η, τ^2), respectively. Find minimal sufficient statistics for the following three cases:

(a) μ, η, σ, τ are arbitrary: $-\infty < \mu, \eta < \infty$, $0 < \sigma, \tau$.

(b) $\sigma = \tau$ and μ, η, σ are arbitrary.

(c) $\mu = \eta$ and μ, σ, τ are arbitrary.