Practice Problems on Maximum Likelihood Estimation

March 20, 2016 STA 5327

1. In a survival time study n cancer patients were observed for a fixed time T after operation and if the symptoms reappear, the time X, since the operation, this happens is recorded. For r of these patients symptoms reappeared at times $x_1, x_2, \ldots, x_r$ after their operation and the remaining $n - r$ patients were still free of symptoms at the end of the time period T. If the time X to the return of symptoms has exponential distribution with p.d.f.

$$f(x \mid \theta) = \theta e^{-\theta x}, \quad x > 0$$

Find the MLE of θ on the basis of the study results.

2. Suppose we take one observation, X, from the discrete distribution, where $0 < \theta < 1$.

Table 1: Probability mass function of X

x	-2	-1	0	1	2
$P(X = x \mid \theta)$	$(1-\theta)/4$	$\theta/12$	$1/2$	$(3-\theta)/12$	$\theta/4$

Find a real-valued statistic $T(X)$ with $\mathbb{E}[T(X)] = \theta$ (T is an unbiased estimator of θ, but might take values outside $[0, 1]$). Obtain the maximum likelihood estimator (MLE) $\hat{\theta}(X)$ of θ and show that it is not unique. Is any choice of MLE unbiased?

3. Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be independent and identically distributed random 2-vectors taking values in the unit square $[0, 1] \times [0, 1]$ with

$$\mathbb{P}(X_1 > x, Y_1 > y) = (1 - x) \cdot (1 - y) \cdot \min\{(1 - x), (1 - y)\}^{\theta}$$

for $0 \leq x \leq 1, 0 \leq y \leq 1$, where $\theta > 0$ is unknown.

   (a) Obtain the likelihood for θ.
   (b) Find the MLE $\hat{\theta}_n(x, y)$ for θ.

4. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{R}^p$ be i.i.d with density

$$f_{\boldsymbol{\theta}}(\mathbf{x}) = c(\alpha) \exp\{- \|\mathbf{x} - \boldsymbol{\theta}\|^{\alpha}\}, \boldsymbol{\theta} \in \mathbb{R}^p, \alpha \geq 1$$

where $c(\alpha) = \int_{\mathbb{R}^p} \exp\{- \|x\|^{\alpha}\} d\mathbf{x}$ and $\|\cdot\|$ is the Euclidean norm.

   (a) Show that if $\alpha > 1$, the MLE $\hat{\boldsymbol{\theta}}$ exists and is unique.
   (b) Show that if $\alpha = 1$, the MLE $\hat{\boldsymbol{\theta}}$ exists and but is not unique if n is even.