September 24, 2013 Debdeep Pati

1 Random variables

1.1 Benefits of using random variables

1. The concept of random variables helps us to formulate our problems better.
2. It allows us to apply all our knowledge of functions to the study of probability.
3. Also, it strips away real world differences to reveal situations that are probabilistically identical.
4. For instance, if X is the number of heads in three flips of a coin and Y is the number of girls in a family with three children, then X and Y are probabilistically identical (assuming the probability is 1/2 of each child being a girl) even though the physical situations they model are quite different.

1.2 Cumulative Distribution Functions

1. Given a discrete r.v. X, the distribution (or c.d.f.) of X is dened by $F(x) = P(X \leq x)$, $x \in \mathbb{R}$.
2. Some properties: $F(-\infty) = 0, F(+\infty) = 1$, F is nondecreasing
3. F is right-continuous: $F(x+) = F(x)$, $\forall x \in \mathbb{R}$
4. If X is discrete, $F(\cdot)$ is a step function
5. Plot of cumulative distribution function: Let X be a random variable taking $-2.4, -1.4, -1.1, -1.0, -0.7, -0.5, 0.1, 0.2, 0.3, 0.8$ with equal probability 1/10.
6. All prob questions can be answered in terms of c.d.f.!
7. $P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$.
8. $P(X < a) = \lim_{a_n \uparrow a} P(X \leq a_n) = \lim_{a_n \uparrow a} F(a_n) = F(a-)$ (left-limit)
9. $P(X = a) = P(X \leq a) - P(X < a) = F(a) - F(a-)$
10. If $F(\cdot)$ is continuous at a, $P(X = a) = F(a) - F(a-) = 0$, since $F(a-) = F(a) = F(a+)$.

11. All prob questions can be answered in terms of c.d.f.! For example,
12. $P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$.
13. $P(X < a) = \lim_{a_n \uparrow a} P(X \leq a_n) = \lim_{a_n \uparrow a} F(a_n) = F(a-)$ (left-limit)
14. $P(X = a) = P(X \leq a) - P(X < a) = F(a) - F(a-)$
15. If $F(\cdot)$ is continuous at a, $P(X = a) = F(a) - F(a-) = 0$, since $F(a-) = F(a) = F(a+)$.

For example, Let

$$F(x) = \begin{cases} 0, x < 0 \\ x/2, 0 \leq x < 1, \\ 2/3, 1 \leq x < 2, \\ 11/12, 2 \leq x < 3 \\ 1, 3 \leq x \end{cases}$$

Find $P(X < 3), P(X = 1), P(X > 1/2), P(2 < X \leq 4)$

$$\begin{aligned} P(X < 3) &= F(3-) = 11/12 \\ P(X = 1) &= F(1) - F(1-) = 2/3 - 1/2 \\ P(X > 1/2) &= 1 - P(X \leq 1/2) = 1 - F(1/2) = 1 - 1/4 \\ P(2 < X \leq 4) &= F(4) - F(2) = 1/12 \end{aligned}$$

1.3 Expectation of a random variable

1. Expected value of X is a weighted average of the possible values that X can take on, each value being weighted by the probability that X assumes it.
2. Suppose we are throwing a fair die. What is the expected number we will see?
3. $E(X) = \sum_{all x} xp(x)$.
4. $E(X)$: weighted average (weights determined by p.m.f. p())
5. Expected value is what one should expect if the experiment is repeated many times.
6. A measure of central tendency (location)

7. p(0) = 1/2, p(1) = 1/2, then E(X) = 0(1/2) + 1(1/2) = 1/2
8. p(0) = 1/3, p(1) = 2/3, then E(X) = 0(1/3) + 1(2/3) = 2/3
9. p(1) = p(2) = p(3) = 1/12, p(4) = P(5) = P(6) = 1/4, then E(X) = 4.25

1.4 Indicator Random Variable

1. Indicator R.V: Given an event $A \subset S$, define $I_A(s) =$

$$\begin{cases} 1, ifs \in A \\ 0, ow \end{cases}$$

2. Then $I_A(\cdot)$is a discrete r.v.
3. Find $E(I_A)$
4. Very useful fact: $P(A) = E(I_A)$.

1.5 Example

A school class of 120 students are driven in 3 buses to a symphonic performance. There are 36 students in one of the buses, 40 in another, and 44 in the third bus. When the buses arrive, one of the 120 students is randomly chosen. Let X denote the number of students on the bus of that randomly chosen student, find E(X).

1. The range of X is 36, 40, 44.
2. P(X = 36) = 36/120, P(X = 40) = 40/120, P(X = 44) = 44/120
3. E(X) = 36(36/120)+40(40/120)+44(44/120) = 40.2667

1.6 Expectation of a function of a random variable

1. Expectation of a function of a random variable
2. We know $E(X) = \sum_x xp(x)$
3. What is E(g(X))? (expectation of a transformed r.v.)
4. Note that g(X) is also a random variable.
5. In fact, any function of a random variable is also a random variable

6. Let $Y = g(X)$. Then $E(g(X)) = E(Y)$.
7. $P(X = -1) = 0.2, P(X = 0) = 0.5, P(X = 1) = 0.3$
8. What is $E(X^2)$?
9. Let $Y = X^2$, still a r.v. since $Y(s) = X(s)X(s), s \in S$.
10. What is the pmf of Y ?
11. $P(Y = 0) = P(X^2 = 0) = P(X = 0) = 0.5, P(Y = 1) = P(X^2 = 1) = P(X = 1 or - 1) = 0.2 + 0.3 = 0.5$
12. $E(Y) = 1 \times 0.5 + 0 \times 0.5 = 0.5$
13. Therefore, $E(X^2) = 0.5$.
14. Note that $E(X^2) \neq E(X)E(X)$.
15. Proposition. If X is a discrete r.v. that takes on one of the values $x_i, , i \geq 1$, with respective probabilities $p(x_i)$ (so p() is a pmf), then for any real-valued function g $E(g(X)) = \sum_i g(x_i)p(x_i)$.
16. For example, $E(X^2) = (-1)^2 \times 0.2 + 1^2 \times 0.3 = 0.5$.
17. Corollary: Corollary. $E(aX + b) = aE(X) + b$, where a and b are constants.