Random Variables - Class 11

September 26, 2013 Debdeep Pati

1 Random variables

1.1 Expectation of a function of a random variable

1. Expectation of a function of a random variable
2. We know $E(X) = \sum_x xp(x)$
   Example: A school class of 120 students are driven in 3 buses to field trip. There are 36 students in one of the buses, 40 in another, and 44 in the third bus. When the buses arrive, one of the 120 students is randomly chosen. Let X denote the number of students on the bus of that randomly chosen student, find E(X).
   (a) The range of X is 36, 40, 44.
   (b) P(X = 36) = 36/120, P(X = 40) = 40/120, P(X = 44) = 44/120
   (c) E(X) = 36(36/120)+40(40/120)+44(44/120) = 40.2667
3. What is E(g(X))? (expectation of a transformed r.v.)
4. Note that g(X) is also a random variable.
5. In fact, any function of a random variable is also a random variable
6. Let $Y = g(X)$. Then $E(g(X)) = E(Y)$.
7. $P(X = -1) = 0.2, P(X = 0) = 0.5, P(X = 1) = 0.3$
8. What is $E(X^2)$?
9. Let $Y = X^2$, still a r.v. since $Y(s) = X(s)X(s), s \in S$.
10. What is the pmf of Y ?
11. $P(Y = 0) = P(X^2 = 0) = P(X = 0) = 0.5, P(Y = 1) = P(X^2 = 1) = P(X = 1 or -1) = 0.2 + 0.3 = 0.5$
12. $E(Y) = 1 \times 0.5 + 0 \times 0.5 = 0.5$
13. Therefore, $E(X^2) = 0.5$.
14. Note that $E(X^2) \neq E(X)E(X)$.

15. Proposition. If X is a discrete r.v. that takes on one of the values $x_i, , i \geq 1$, with respective probabilities $p(x_i)$ (so p() is a pmf), then for any real-valued function g $E(g(X)) = \sum_i g(x_i)p(x_i)$.
16. For example, $E(X^2) = (-1)^2 \times 0.2 + 1^2 \times 0.3 = 0.5$.
17. Corollary: Corollary. $E(aX + b) = aE(X) + b$, where a and b are constants.
18. Moments and Variance (raw moments and central moments) kth raw moment: $E(X^k)$, kth central moment: $E((X - E(X))^k)$
19. Some propositions. $Var(X) = E(X^2) - (EX)^2$, $Var(b + aX) = a^2 Var(X)$, where a, b are constants.
20. What is Var(X) for the former r.v. X? ($P(X = -1) = 0.2, P(X = 0) = 0.5, P(X = 1) = 0.3$)
21. What is $Var(2X)$?
22. $E(X)$: a measure of the location of X
23. $Var(X)$: a measure of the spread (scale) of X
24. $SD(X) = \sqrt{Var(X)}$: the standard deviation of X

2 Commonly used random variables

2.1 Bernoulli Random variable

1. Example. Flip a coin. If the outcome is heads, $X = 1$, otherwise $X = 0$.
2. Then X is a Bernoulli r.v., or X follows a Bernoulli distribution.
3. Denition. $P(X = 1) = p, P(X = 0) = 1?p$, where $p \in (0, 1)$.
4. Then $X \sim Bernoulli(p)$
5. $E(X) = p$, $Var(X) = p(1 - p)$

2.2 Binomial Random Variable

1. Example. Flip five coins independently. X : the number of heads. Find the p.m.f. of X.
2. $P(X = 0)$

3. $P(X = k)$

4. Consider a general problem: Perform n independent trials each of which results in a success w.p. p and in a failure w.p. 1 - p. Let X represent the number of successes that occur in these n trials. What is the pmf of X?

5. X is a binomial random variable with parameters (n, p).

2.2.1 Properties

1. Let $X \sim Bin(n, p)$ with $0 < p < 1$. Then as k goes from $0 \to n$, $P(X = k)$ first increases monotonically then decreases monotonically, reaching its largest value when k is the largest integer $\leq (n+1)p$. $\frac{p(k)}{p(k-1)} = \frac{(n-k+1)p}{k(1-p)}$

2. Discuss skewed binomial distribution

3. Let $X \sim B(n, p)$. $E(X) = np, Var(X) = np(1-p)$ Use the binomial theorem! See Ross (P139).

Problem

It is known that screws produced by a certain company will be defective with probability .01 independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

If X is the number of defective screws, then X is a binomial random variable with parameters $(10, .01)$. The probability that a package will have to be replaced is $1 - P(X = 0) - P(X = 1)$

Problem

In a gambling game, a player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, i = 1,2,3, then the player wins i units; on the other hand, if the number bet by the player does not appear on any of the dice, then the player loses 1 unit, Is this game fair to the player? If we assume that the dice are fair and act independently of each other, then the number of times that the number bet appears is a binomial random variable with parameter (3, 1/6) Let X denote the players winnings, we have

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\begin{aligned}
P(X = -1) &= bin(3, 1/6, 0) = 125/216, P(X = 1) = bin(3, 1/6, 1) = 75/216, \\
P(X = 2) &= bin(3, 1/6, 2) = 15/216, P(X = 3) = bin(3, 1/6, 3) = 1/216,
\end{aligned}
$$

$E(X) = -17/216.$