Jointly distributed random variables

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1 Mixture of continuous and discrete

$X \sim Beta(a, b)$ for parameters $a, b > 0$ is the pdf is given by

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\int_0^1 x^{a-1}(1-x)^{b-1}dx}, 0 < x < 1$$

The normalizing constant $\int_0^1 x^{a-1}(1-x)^{b-1}dx$ is also denoted by $Beta(a, b)$.

X is a continuous random variable having probability density function f; N is a discrete random variable. Then

$$f(X = x \mid N = n) = \frac{P(X = x, N = n)}{P(N = n)} = f_X(x)\frac{P(N = n \mid X = x)}{P(N = n)}$$

1. Consider $n+m$ trials having a common probability of success. Suppose, however, that this success probability is not fixed in advance but is chosen from a $uniform(0, 1)$ population. What is the conditional distribution of the success probability given that the $n+m$ trials result in n successes? Let X denote the trial success probability, which is $U(0, 1)$. N denote the number of successes, which is $B(n + m, x)$ because $n+m$ trials are independent given $X = x$. The conditional density of X given $N = n$ is

$$\begin{aligned} f_{X|N}(x \mid n) &= \frac{P(N = n \mid X = x)f_X(x)}{P(N = n)} \\ &= \frac{\binom{n+m}{n}x^n(1-x)^m}{P(N = n)} \\ &= \frac{\binom{n+m}{n}x^n(1-x)^m}{\int_0^1 \binom{n+m}{n}x^n(1-x)^m dx} \\ &= \frac{x^n(1-x)^m}{\int_0^1 x^n(1-x)^m dx} \end{aligned}$$

Thus $X \mid N = n \sim Beta(n + 1, m + 1)$.

2 Chapter 7: Properties of Expectation

The expected value of a discrete random variable X is defined by

$$E(X) = \sum_{allx} xp(x)$$

For continuous random variables:

$$E(X) = \int xf(x)dx$$

If $P(a \leq X \leq b) = 1$, then $a \leq E[X] \leq b$.

3 Expectation of functions of multiple random variables

If (X, Y) have a joint probability mass function, then

$$E(g(X, Y)) = \sum_{y} \sum_{x} g(x, y)p(x, y)$$

If X and Y have a joint probability density function, then

$$E(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy$$

3.1 Problem

An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of the accident an ambulance is at a location Y that is also uniformly distributed on the road. Assuming that X and Y are independent, find the expected distance between the ambulance and the point of the accident. $E(|X - Y| =?$

Clearly, $f(x, y) = 1/L^2, 0 < x < l, 0 < y < L$.

$$E|X - Y| = \frac{1}{L^2} \int_{x=0}^{L} \int_{y=0}^{L} |x - y|dxdy = \frac{L}{3}$$

3.2 Properties of Expectation

1. $E(X + Y) = E(X) + E(Y)$ for both discrete and continuous random variables.
2. Suppose that for random variables X and Y, $X \geq Y, X - Y \geq 0, E[X - Y] \geq 0 E[X] \geq E[Y]$.

3. If $E[X_i]$ is finite for all $i = 1, \ldots, n$, then $E[X_1 + \ldots + X_n] = E[X_1] + \ldots + E[X_n]$.

4. The sample mean Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having distribution function F and expected value μ. Such a sequence of random variables is said to constitute a sample from the distribution F. The quantity $\bar{X}$, defined by

$$\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$$

is called sample mean. $E(\bar{X}) = \sum_{i=1} n \frac{E(X_1 + E(X_2) + \cdots E(X_n)}{n} = \frac{n\mu}{n} = \mu$.

Example: (Saint Petersburg Paradox) A casino offers a game of chance for a single player in whicha fair coin is tossed at each stage. The pot starts at 1 dollar and is doubled every time a head appears. The first time a tail appears, the game ends and the player wins whatever is in the pot. Thus the player wins 1 dollar if a tail appears on the first toss, 2 dollars if a head appears on the first toss and a tail on the second, 4 dollars if a head appears on the first two tosses and a tail on the third, 8 dollars if a head appears on the first three tosses and a tail on the fourth, and so on. In short, the player wins 2^{k-1}dollars if the coin is tossed k times until the first tail appears. What is the expected payout? ($\sum_{k=1}^{\infty} 2^{k-1} \frac{1}{2^{k-1}} \frac{1}{2} = \infty$)

Boole's Ineq: Let $A_1, A_2, \ldots, A_n$ denote the events and define the indicator variables $X_i, i = 1, \ldots, n$ by

$$X_i = \begin{cases} 1, \; if \; A_i occurs \\ 0, \; otherwise \end{cases}$$

let $X = \sum_{i=1}^{n} X_i$. SO X is the number of events A_i that occurs. Define

$$Y = \begin{cases} 1, \; if \; X \geq 1 \\ 0, \; otherwise \end{cases}$$

Hence $Y = 1$ if at least one of the A_i occurs and is 0 otherwise. From the fact $X \geq Y$ and hence $E(X) \geq E(Y)$ we obtain the famous Boole's inequality

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$$

5. Expectation of a Binomial random variable X with parameter n and p represents the number of successes in n independent trials when each trial has probability p of being

a success. Let $X \sim Bin(n, p)$. Since X represents the number of successes in n trials, $X = X_1 + X_2 + \ldots + X_n$, where

$$X_i = \begin{cases} 1, \text{ if the ith trial results in a success} \\ 0, \text{ if the ith trial results in a failure} \end{cases}$$

Clearly, $X_i \sim Bernoulli(p)$ so that $E(X_i) = p$ and hence $E(X) = \sum_{i=1}^{n} E(X_i) = np$.