1   Problem

An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of the accident an ambulance is at a location Y that is also uniformly distributed on the road. Assuming that X and Y are independent, find the expression for the expected distance between the ambulance and the point of the accident. $(E(|X - Y|))$

2   Properties of expectation

1. Mean of a hypergeometric random variable If n balls are randomly selected from an urn containing N balls of which m are white, find the expected number of white balls selected. Let X denote the number of white balls selected, $X = X_1 + \cdots + X_m$ where

$$X_i = \begin{cases} 1, \text{ if the ith white ball is selected} \\ 0, \text{ otherwise} \end{cases}$$

$E(X_i) = P(X_i = 1) = \frac{\binom{1}{1}\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N}$ Hence $E(X) = mn/N$.

Alternatively, $X = Y_1 + \cdots + Y_n$ where

$$Y_i = \begin{cases} 1, \text{ if the ith ball selected is white} \\ 0, \text{ otherwise} \end{cases}$$

Since the i ball selected is equally likely to be any of the N balls

$$E(Y_i) = \frac{m}{N}$$

So $E(X) = mn/N$

2. A group of N people throw their hats into the center of a room. The hats are mixed up, and each person randomly selects one. Find the expected number of people that select their own hat. Let X denote the number of matches, we can $E(X)$ by $X = X_1 + X_2 + \ldots X_n$ where

$$X_i = \begin{cases} 1 \text{ if the ith person selects his own hat} \\ 0 \text{ otherwise} \end{cases}$$

Since, for each i, the ith person is equally likely to select any of the N hats

$$E(X_i) = P(X_i = 1) = \frac{1}{N}$$

Thus

$$E(X) = E(X_1) + \ldots + E(X_N) = \frac{1}{N} \times N = 1$$

On an average, exactly one person selects his own hat.

3. Suppose that there are N different types of coupons and each time one obtains a coupon it is equally likely to be any one of the N types. Find the expected number of coupons one need amass before obtaining a complete set of at least one of each type.

   Let X denote the number of coupons collected before a complete set is attained. We define $X_i, i = 0, 1, \ldots .N - 1$ to be the number of additional coupons that need to be obtained after i distinct types have ten collected in order to obtain another type, and note that $X = X_0 + X_1 + \ldots, X_{N-1}$. When i distinct types of coupons have already been collected, a new coupon obtained will be of a distinct type with probability $(N - i)/N$. Hence

   $$P(X_i = k) = \frac{N - i}{N} \left(\frac{i}{N} \right)^{k-1}, k \geq 1$$

   Clearly, X_i is a geometric random variable with parameter $(N-i)/N$. Hence $E(X_i) = \frac{N}{N-i}$ Thus

   $$E(X) = 1 + \frac{N}{N - 1} + \frac{N}{N - 2} + \ldots + N = N \left(1 + \ldots + \frac{1}{N - 1} + \frac{1}{N} \right)$$

2.1   Moments of the number of Events that occur

1. Concept of moment is evolved from the moment concept in physics.

2. The nth moment (about zero) of a probability density function f(x) is the expected value of $E(X^n)$.

3. For discrete random variables with probability mass function $p(x)$. $E(X^n) = \sum_{allx} x^n p(x)$

4. For continuos random variables with probability density function $f(x)$. $E(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx$

5. $E[X]$ is the first moment of X, $E[X^2]$ is the second moment of the pdf of X and so on.

6. For given events $A_1, \ldots, A_n$, find $E[X]$, where X is the number of these events that occur. We define an indicator variable I_i for event A_i, and $I_i = 1$ if A_i occurs. As $X = \sum_{i=1}^{n} I_i$, we have $E(X) = E(\sum_{i=1}^{n} I_i) = \sum_{i=1}^{n} E(I_i) = \sum_{i=1}^{n} P(A_i)$.

7. We are interested in the number of pairs of events that occurs, the number of pairs is equal to $\sum_{i<j} I_i I_j$, which is equal to $\binom{X}{2}$. Hence

$$E\left[\binom{X}{2}\right] = \sum_{i<j} E(I_i I_j) = \sum_{i<j} P(A_i \cap A_j)$$

$$E(X^2 - X) = 2 \sum_{i<j} P(A_i \cap A_j)$$

8.

$$\binom{X}{k} = \sum_{i_1 < i_2 < \ldots i_k} I_{i_1} I_{i_2} \ldots I_{i_k}$$

$$E\left[\binom{X}{k}\right] = \sum_{i_1 < i_2 < \cdots < i_k} E(I_{i_1} I_{i_2} \ldots I_{i_k}) = \sum_{i_1 < i_2 < \ldots < i_k} P(A_{i_1} A_{i_2} \ldots \cap A_{i_k})$$

2.2 Moments of Binomial random variable

Consider n independent trials, with each trial being a success with probability p. Let A_i be he event that the trial i is a success. When $i \neq j$, $P(A_i \cap A_j) = p^2$. Clearly $E(\binom{X}{2}) = \binom{n}{2}p^2$ and hence $E(X^2) = n(n-1p^2 + np.$ $E(X(X-1)(X-2)) = n(n-1)(n-2)p^3$ and hence $E(X^3) = 3E(X^2) - 2E(X) + n(n-1)(n-2)p^3 = 3n(n-1)p^2 + np + n(n-1)(n-2)p^3.$