1 Covariance, Variance of Sums and Correlations

Theorem 1 *If X and Y are independent, then for any function h and g*

$$E(g(X)h(Y)) = E(g(X))E(h(Y))$$

The covariance between X and Y, denoted by $Cov(X,Y)$, is defined by

$$\begin{aligned} Cov(X,Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY - E[X]Y - XE[Y] + E[Y]E[X]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

If X and Y are independent, $Cov(X,Y) = 0$. If $Cov(X,Y) = 0$, X and Y may not be independent. $P(X = 0) = P(X = 1) = P(X = -1) = 1/3$ $Y = 0$ if $X \neq 0$, $Y = 1$ if $X = 0$ $XY = 0$, $E[XY] = 0$, and $E[X] = 0$, so $Cov(X,Y) = E[XY] - E[X]E[Y] = 0$ But X and Y are not independent.

Theorem 2

1. *$Cov(X,Y) = Cov(Y,X)$*
2. *$Cov(X,X) = Var(X)$*
3. *$Cov(aX,Y) = aCov(X,Y)$*
4. *$Cov(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) = \sum_{i=1}^n \sum_{j=1}^m Cov(X_i, X_j)$*

Hence

$$\begin{aligned} Var(\sum_{i=1}^n X_i) &= Cov(\sum_{i=1}^n, \sum_{i=1}^n X_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n Cov(X_i, X_j) \\ &= \sum_{i=1}^n Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j) \end{aligned}$$

which is equivalent to the following

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

If $X_1, \ldots, X_n$ are pairwise independent, we have

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$$

1.1 Correlation of two random variables

Correlation of two random variables X and Y , denoted by $\rho(X, Y)$, and is defined by

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

Theorem 3 $-1 \leq \rho(X, Y) \leq 1$

$$\begin{aligned} 0 &\leq Var(X/\sigma_x + Y/\sigma_y) \\ &= 2(1 + \rho(X, Y)) \end{aligned}$$

Showing that $\rho(X, Y) \geq -1$ and using

$$\begin{aligned} 0 &\leq Var(X/\sigma_x - Y/\sigma_y) \\ &= 2(1 - \rho(X, Y)) \end{aligned}$$

showing that $\rho(X, Y) \leq 1$.

Example: Let I_A and I_B be indicator variables for the event A and B respectively. Then

$$\begin{aligned} Cov(I_A, I_B) &= P(A \cap B) - P(A)P(B) \\ &= P(A \mid B)P(B) - P(A)P(B) \\ P(B)[P(A \mid B - P(A)] & \end{aligned}$$