1 Stochastic Block models

Let $A = (A_{ij}) \in \{0,1\}^{n\times n}$ denote the adjacency matrix of a network with n nodes, with $A_{ij} = 1$ indicating the presence of an edge from node i to node j and $A_{ij} = 0$ indicating a lack thereof. We will consider directed networks without self-loops so that A_{ij} and A_{ji} need not be the same and $A_{ii} = 0$.

$$Q_{rs} \overset{\text{ind}}{\sim} U(0,1), \quad r,s = 1,\dots,k, \tag{1}$$

$$P(z_i = k \mid \pi) = \pi_k, \quad i = 1,\dots,n, \tag{2}$$

$$\pi \sim \text{Dirichlet}(\alpha_1,\dots,\alpha_k), \tag{3}$$

$$A_{ij} \mid z, Q \overset{\text{ind}}{\sim} \text{Bernoulli}(\theta_{ij}), \quad \theta_{ij} = Q_{z_i z_j}. \tag{4}$$

A hierarchical specification as in (or very similar to) (1) – (4) has been commonly used in the literature; see for example, [4, 3, 1, 2]. Analytic marginalizations can be carried out due to the conjugate nature of the prior, facilitating posterior sampling [2]. In particular, using standard multinomial-Dirichlet conjugacy, the marginal prior of z can be written as

$$p(z) = \frac{\Gamma(\sum_{r=1}^k \alpha_r)}{\Gamma(n + \sum_{r=1}^k \alpha_r)} \prod_{r=1}^k \frac{\Gamma(n_r + \alpha_r)}{\Gamma(\alpha_r)}, \quad z \in \mathcal{Z}_{n,k}, \tag{5}$$

where recall that $n_r = \sum_{i=1}^n \mathbb{1}(z_i = r)$.

2 Gibbs sampling for fixed k

Define

$$\begin{aligned}
n_r &= \sum_{i=1}^n I(z_i = r), \quad r = 1,\dots,k. \\
n_{rs} &= \sum\sum_{1\le i\ne j\le n} I(z_i = r, z_j = s) = n_r n_s - n_r I(r = s). \\
A[rs] &= \sum\sum_{(i,j): z_i = r, z_j = s} A_{ij}, \quad r = 1,\dots,k, s = 1,\dots,k.
\end{aligned}$$

Then we have

$$\begin{aligned} \pi \mid - &\sim \text{Dirichlet}(\alpha_1 + n_1, \ldots, \alpha_k + n_k) \\ Q_{rs} \mid - &\sim \text{Beta}(1 + A[rs], 1 + n_{rs} - A[rs]). \end{aligned}$$

Note that

$$P(z_i = l \mid z_{-i}, A, \pi, Q) \propto P(A \mid z, \pi, Q) P(z \mid \pi) P(\pi) P(Q).$$

Keeping the terms involving z_i,

$$P(A \mid z, \pi, Q) \propto \left\{ \prod_{j \neq i} Q_{z_i z_j}^{A_{ij}} (1 - Q_{z_i z_j})^{1 - A_{ij}} \right\} \times \left\{ \prod_{k \neq i} Q_{z_k z_i}^{A_{ki}} (1 - Q_{z_k z_i})^{1 - A_{ki}} \right\}, \quad P(z \mid \pi) \propto \pi_{z_i}.$$

Hence,

$$P(z_i = l \mid z_{-i}, A, \pi, Q) \propto \pi_{z_i} \times \left\{ \prod_{j \neq i} Q_{z_i z_j}^{A_{ij}} (1 - Q_{z_i z_j})^{1 - A_{ij}} \right\} \times \left\{ \prod_{k \neq i} Q_{z_k z_i}^{A_{ki}} (1 - Q_{z_k z_i})^{1 - A_{ki}} \right\}.$$

References

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