Sample Space, events and probability - Class 4

September 5, 2013 Debdeep Pati

Example on properties of probability:

1. J is taking two books along on her holiday vacation. With probability .5 she will like the first book; with probability .4 she will like the second book; with probability .3 she will like both book. What is the probability she like neither book?
2. A total of 36 members of a club play tennis, 28 play squash, and 18 play badminton. Furthermore, 22 of the members play both tennis and squash, 12 play both tennis and badminton, 9 play both squash and badminton, and 4 play all three sports. How many members of this club play at least one of these sports?

1 Inclusion-exclusion principle

1. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
2. Inclusion Exclusion principle:

$$P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{1 \le i < j \le n} P(E_i \cap E_j) + \sum_{1 \le i < j < k \le n} P(E_i \cap E_j \cap E_k) + \qquad (1)$$

$$(-1)^{r+1} \sum_{1 \le i_1 < i_2 < i_3 < \ldots i_r \le n} P(E_{i_1} \cap E_{i_2} \cap E_{i_r}) + (-1)^{n+1} P(E_1 \cap \cdots E_n) \qquad (2)$$

The summation is taken over all possible $\binom{n}{r}$ many possible subsets of size r of the set $\{1, 2, \ldots, r\}$.

The name comes from the idea that the principle is based on over-generous inclusion, followed by compensating exclusion. When $n > 2$ the exclusion of the pairwise intersections is (possibly) too severe, and the correct formula is as shown with alternating signs. This formula is attributed to Abraham de Moivre; it is sometimes also named for Daniel da Silva, Joseph Sylvester or Henri Poincare.

2 Sample space having equally likely outcomes

For $S = \{1, 2, \ldots, N\}$, if $P(\{1\}) = P(\{2\}) = \ = P(\{N\})$, then $P(\{i\}) = 1/N$. For any event E, $P(E)$ = (number of outcomes in E) / (number of outcomes in S). Probability of any events equals the proportion of outcomes in the sample space that are contained in E.

Examples

1. If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

   There are 6 possible outcomes, namely $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$, that give the sum of 7. Since there are totally 36 outcomes, the desired probability is $1/6$.

2. If 3 balls are randomly drawn from a bowl containing 6 white and 5 black balls, what is the probability that one of the drawn balls is white and the other two black?

$$\frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}}$$

3. An urn contains n balls, of which one is special. If k of these balls are withdrawn one at a time, which each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

4. Suppose that $n+m$ balls, of which n are red and m are blue, are arranged in a linear order in such a way that all $(n+m)!$ possible orderings are equally likely. If we record the result of this experiment by only listing the colors of the successive balls, show that all the possible results remain equally likely.

5. **Birthday paradox:** If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than 1/2?$(365(365-1)\dots(365-n+1)/365^n$ When $n \geq 23$, this probability is less than 0.5. What is the probability that a pair will have the same birthday $(365/(365)^2)$

6. **Pennsylvania lottery:** Choose 7 numbers from 1 through 80. The state chooses 11, randomly. You win if all 7 of your numbers are among the state's 11. What are your chances of winning? (The state can pick the 11 numbers in $\binom{80}{11}$ ways so there are $\binom{11}{7}$ possible winning choices of 7 numbers. There are $\binom{80}{7}$ overall choices, so your chances of winning are $\binom{11}{7}/\binom{80}{7}$)

7. **Card problems:** We pick 5 cards at random from a standard deck of 52 playing cards. There are thirteen values, from 2 to 10, and then J, K, Q, and A. Each value has four cards, one each of Spades, Hearts, Clubs, and Diamonds.

   (a) What is the probability that you will get four cards of the same value? $(13.\binom{4}{4}.48/\binom{52}{5})$

   (b) What is the probability that you will get three cards of one value and two of another value? $(13.12.\binom{4}{2}\binom{4}{3}/\binom{52}{5})$

(c) What is the probability that you will get three cards of one value, but not have one of the combinations in (a) and (b)? $(13.\binom{4}{3}\binom{48}{1}\binom{44}{1}/\binom{52}{5})$

(d) A poker hand consists of 5 cards. If the cards have distinct consecutive values and are not all of the same suit, we say that the hand is a straight. For instance, a hand consisting of the five of spades, six of spades, seven of spades, eight of spades, and nine of hearts is a straight. What is the probability that one is dealt a straight?

$$\frac{10(4^5 - 4)}{\binom{52}{5}}$$